PROBLEMS AND SOLUTIONS

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Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before February 28, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11523. *Proposed by Timothy Chow, Princeton, NJ.* Given boxes 1 through n, put balls in k randomly chosen boxes. The *score* of a permutation π of $\{1, \dots, n\}$ is the least i such that box $\pi(i)$ has a ball. Thus, if $\pi = (3, 4, 1, 5, 2)$ with $(n, k) = (5, 2)$, and boxes 1 and 4 have balls, then π has score 2.
(**a**) A permutation π is *fair* if, regardless of the value of k, the probability that π scores lower than the identity permutation equals the probability that it scores higher. Show that π is fair if and only if for each i in $[1, n]$, either $\pi(i) > i$ and $\pi^{-1}(i) > i$, or $\pi(i) \le i$ and $\pi^{-1}(i) \le i$.
(**b**) Let $f(n)$ be the number of fair permutations of $\{1, \dots, n\}$, with the convention that $f(0) = 1$. Show that $\sum_{n=0}^{\infty} f(n)x^n/n! = e^x \sec(x)$.
(**c**) Assume now that $n = m^3$ with $m \ge 2$, and the boxes are arranged in m rows of length m^2. Alice scans the top row left to right, then the row below it, and so on, until she finds a box with a ball in it. Bob scans the leftmost column top to bottom, then the next column, and so on. They start simultaneously and both check one box per second. For which k are Alice and Bob equally likely to be the first to discover a ball?

11524. *Proposed by H. A. ShahAli, Tehran, Iran.* A vector v in $\mathbb{R}^n$ is *short* if $\|v\| \le 1$.
(**a**) Given six short vectors in $\mathbb{R}^2$ that sum to zero, show that some three of them have a short sum.
(**b**)* Let $f(n)$ be the least M such that, for any finite set T of short vectors in $\mathbb{R}^n$ that sum to 0, and any integer k with $1 \le k \le |T|$, there is a k-element subset S of T such that $\|\sum_{v \in S} v\| \le M$. The result of part (**a**) suggests $f(2) = 1$. Find $f(n)$ for $n \ge 2$.

11525. *Proposed by Grigory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.*
(**a**) Prove that for each $n \ge 3$ there is a set of regular n-gons in the plane such that every line contains a side of exactly one polygon from this set.

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(**b**) Is there a set of circles in the plane such that every line in the plane is tangent to exactly one circle from the set?
(**c**) Is there a set of circles in the plane such that every line in the plane is tangent to exactly two circles from the set?
(**d**) Is there a set of circles in the plane such that every line in the plane is tangent to exactly three circles from the set?

11526. *Proposed by Marius Cavachi, "Ovidius" University of Constanta, Constanta, Romania.* Prove that there is no function f from $\mathbb{R}^3$ to $\mathbb{R}^2$ with the property that $\|f(x) - f(y)\| \geq \|x - y\|$ for all $x, y \in \mathbb{R}^3$.

11527. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.* Prove that in an acute triangle with sides of length a, b, c, inradius r, and circumradius R,

$$\frac{a^2}{b^2 + c^2 - a^2} + \frac{b^2}{c^2 + a^2 - b^2} + \frac{c^2}{a^2 + b^2 - c^2} \geq \frac{3}{2} \cdot \frac{R}{r}.$$

11528. *Proposed by Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let p, a, and b be positive integers with $a < b$. Consider a sequence $\langle x_n \rangle$ defined by the recurrence $nx_{n+1} = (n + 1/p)x_n$ and an initial condition $x_1 \neq 0$. Evaluate

$$\lim_{n\to\infty} \frac{x_{an} + x_{an+1} + \cdots + x_{bn}}{nx_{an}}.$$

11529. *Proposed by Walter Blumberg, Coral Springs, FL.* For $n \geq 1$, let $A_n = \left(3\sum_{k=1}^{n} \left\lfloor \frac{k^2}{n} \right\rfloor\right) - n^2$. Let p and q be distinct primes with $p \equiv q \pmod 4$. Show that $A_{pq} = A_p + A_q - 2$.

SOLUTIONS

Splitting Elements of Set Systems

11372 [2008, 568]. *Proposed by Jennifer Vandenbussche and Douglas B. West, University of Illinois at Urbana-Champaign, Urbana, IL.* In a family of finite sets, let a *splitting element* be an element that belongs to at least two of the sets and is omitted by at least two of the sets. Determine the maximum size of a family of subsets of $\{1, \ldots, n\}$ for which there is no splitting element.

Solution by David Gove, California State University, Bakersfield, CA. The maximum size is $n + 1$. Consider a largest such family. Removing x from the sets it lies in and adding it to the others yields another such family. Hence we may assume that each element appears in at most one of the sets. If any of the sets has more than one element, then we can obtain a bigger family by replacing that set by its singleton subsets. Thus the family consisting of the empty set and the singleton sets is a largest such family.

Editorial comment. By the argument above, there are 2^n extremal families. Marian Tetiva sent a thorough discussion of a more general problem. Let $g_s(n)$ be the maximum size of a family of subsets of $\{1, \ldots, n\}$ such that every element appears in at most s sets or avoids at most s sets; the stated problem is $g_1(n) = n + 1$, and clearly $g_0(n) = 1$. By the complementation argument above, we may equivalently seek the largest family such that every element appears in at most s sets. Tetiva proved a bound and conjectured equality. Intuitively, the idea is that one should take all the small sets until the bound on the number of appearances of each element is reached. For example, if $s = \sum_{i=1}^{k} \binom{n-1}{i-1}$, then one should take all the sets of size at most k. When s is not of this form, the exact solution is more difficult.

Also solved by D. Beckwith, P. Corn, D. L. Craft, C. Curtis, P. P. Dályay (Hungary), K. David & P. Fricano, D. Degiorgi (Switzerland), J. Gately, J. Guerreiro (Portugal), H. S. Hwang (Korea), K. Kneile, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), M. D. Meyerson, J. H. Nieto (Venezuela), R. E. Prather, T. Rucker, V. Rutherfoord, K. Schilling, E. Schmeichel, B. Schmuland (Canada), R. Stong, J. Swenson, M. Tetiva (Romania), B. Tomper, Fisher Problem Group, Szeged Problem Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), Houghton College Problem Solving Group, Microsoft Research Problems Group, NSA Problems Group, and the proposers.

A Determinant Generated by a Polynomial

11377 [2008, 664]. *Proposed by Christopher Hillar, Texas A&M University, College Station, TX and Lionel Levine, Massachusetts Institute of Technology, Cambridge, MA.* Given a monic polynomial p of degree n with complex coefficients, let A_p be the $(n+1) \times (n+1)$ matrix with $p(-i+j)$ in position (i, j), and let D_p be the determinant of A_p. Show that D_p depends only on n, and find its value in terms of n.

Solution by John H. Lindsey II, Cambridge, MA. The value of D_p is $(n!)^{n+1}$, which we prove by induction on n. The result is trivial when $n = 0$. For $n > 0$, use indices $0, \dots, n$ for the rows and columns of A_p. In A_p, let C_j be column j and R_i be row i. Given a function f, define Δf by $\Delta f(k) = f(k+1) - f(k)$. By induction on n, if f is a monic polynomial of degree n, then $\Delta^n f(x) = n!$ for all x.

Replacing C_n with $\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} C_j$ does not change the determinant, but it turns the ith entry of column n into $\Delta^n p(-i)$, which equals $n!$. Now for $0 \le i \le n-1$ in order, subtract the next row from R_i, replacing R_i with $R_i - R_{i+1}$. This puts 0 in the last column, except for the last row. For $j < n$, the new entry $a'_{i,j}$ is $p(-i+j) - p(-i-1+j)$, which equals $\Delta p(-i-1+j)$. Since Δp has leading coefficient n, the upper left $(n-1)$-by-$(n-1)$ block has the form nA_f, where $f(x) = (1/n)\Delta p(x-1)$.

Since f is a monic polynomial with degree $n-1$, by the induction hypothesis $D_f = (n-1)!^n$. Expanding the altered D_p down the last column yields $D_p = n!\, n^n (n-1)!^n = n!^{n+1}$.

Editorial comment. Solvers used a variety of methods, including Vandermonde determinants. Roger Horn proved a substantial generalization. Given a matrix A, let $p(A)$ denote the entrywise application of the polynomial p to A; that is, the (i, j)-entry of $p(A)$ is $p(a_{i,j})$. For $x \in \mathbb{C}^{n+1}$, let $A(x)$ be the matrix given by $a_{i,j} = x_i + j - 1$. If p is a monic polynomial of degree n, then

$$\det p(A(x)) = \left(\prod_{i>j} (x_i - x_j) \right) \frac{(-1)^{\lfloor (n+1)/2 \rfloor (n!)^n}}{\prod_{i=1}^{n-1} i!}, \tag{1}$$

which depends only on x and n, not p. The originally stated problem is the case $x = (0, -1, \dots, -n)^T$.

Also solved by D. Beckwith, R. Chapman (U. K.), P. Corn, P. P. Dályay (Hungary), J. Grivaux (France), J. Hartman, C. C. Heckman, R. A. Horn, R. Howard, G. Keselman, O. Kouba (Syria), S. C. Locke, O. P. Lossers (Netherlands), K. McInturff, J. H. Nieto (Venezuela), É. Pité (France), C. R. Pranesachar (India), M. A. Prasad (India), N. C. Singer, J. H. Smith, A. Stadler (Switzerland), V. Stakhovsky, R. Stong, T. Tam, M. Tetiva (Romania), B. Tomper, M. Vowe (Switzerland), L. Zhou, BSI Problems Group (Germany), FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

The Column Space of a Very Nilpotent Matrix

11379 [2008, 664]. *Proposed by Oskar Maria Baksalary, Adam Mickiewicz University, Poznań, Poland, and Götz Trenkler, Technische Universität Dortmund, Dortmund, Germany.* Let A be a complex matrix of order n whose square is the zero matrix. Show that $\mathcal{R}(A + A^*) = \mathcal{R}(A) + \mathcal{R}(A^*)$, where $\mathcal{R}(\cdot)$ denotes the column space of a matrix argument.

Solution by M. Andreoli, Miami Dade College, Miami, FL. Note first that $A^2 = 0$ implies $\mathcal{R}(A) \subseteq \mathcal{N}(A)$, where $\mathcal{N}(A)$ is the nullspace of A. This holds because $y = Ax$ implies $Ay = A^2x = 0$.

For $y \in \mathcal{R}(A + A^*)$, there exists x such that $y = (A + A^*)x = Ax + A^*x$. Hence $y \in \mathcal{R}(A) + \mathcal{R}(A^*)$, and we conclude that $\mathcal{R}(A + A^*) \subseteq \mathcal{R}(A) + \mathcal{R}(A^*)$.

Conversely, for $y \in \mathcal{R}(A) + \mathcal{R}(A^*)$, there exist vectors x_1 and x_2 such that $y = Ax_1 + A^*x_2$. Since $\mathcal{N}(A^*)$ and $\mathcal{R}(A)$ are orthogonal complements in $\mathbb{C}^n$, there exist vectors $u \in \mathcal{R}(A)$ and $v \in \mathcal{N}(A^*)$ such that $x_1 - x_2 = u + v$. Since $\mathcal{R}(A) \subseteq \mathcal{N}(A)$, we have $u \in \mathcal{N}(A)$. Letting $x = x_1 - u = x_2 + v$, we have

$$\begin{aligned}(A + A^*)x = Ax + A^*x \;&=\; A(x_1 - u) + A^*(x_2 + v)\\ &= Ax_1 - Au + A^*x_2 + A^*v \;=\; Ax_1 + A^*x_2 \;=\; y.\end{aligned}$$

Thus $y \in \mathcal{R}(A + A^*)$, and hence $\mathcal{R}(A) + \mathcal{R}(A^*) \subseteq \mathcal{R}(A + A^*)$.

Also solved by M. Bataille (France), P. Budney, R. Chapman (U. K.), P. Corn, C.-K. Fok, J. Freeman, J.-P. Grivaux (France), J. Hartman, E. A. Herman, R. A. Horn, O. Kouba (Syria), C. Lanski, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (Germany), I. Pinelis, É. Pité (France), N. C. Singer, J. H. Smith, R. Stong, J. Stuart, F. Vrabec (Austria), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposers.

A Generalized Binomial Coefficient

11380 [2008, 665]. *Proposed by Hugh Montgomery, University of Michigan, Ann Arbor, MI, and Harold S. Shapiro, Royal Institute of Technology, Stockholm, Sweden.* For $x \in \mathbb{R}$, let $\binom{x}{k} = \frac{1}{k!}\prod_{j=0}^{k-1}(x - j)$. For $k \geq 1$, let a_k be the numerator and q_k the denominator of the rational number $\binom{-1/3}{k}$ expressed as a reduced fraction with $q_k > 0$.
(**a**) Show that q_k is a power of 3.
(**b**) Show that a_k is odd if and only if k is a sum of distinct powers of 4.

Solution by Stephen M. Gagola Jr., Kent State University, Kent, OH. We prove more generally that if $m > 1$ and $m + 1$ is a power of a prime p, and the rational number $\binom{-1/m}{k}$ has numerator a_k and denominator q_k in lowest terms with $q_k > 0$, then
(**a**′) all prime factors of q_k divide m, and
(**b**′) $p \nmid a_k$ if and only if k is a sum of distinct powers of $m + 1$.

The stated problem is the case $m = 3$, where $p = 2$.
(**a**′) For clarity, let $c_k = \binom{-1/m}{k} = a_k/q_k$. In the formal power series ring $\mathbb{Q}[[x]]$,

$$(1 + x)^{-1/m} = \sum_{k=0}^{\infty} c_k x^k. \tag{1}$$

Therefore,

$$\sum_{k=0}^{\infty}(-1)^k x^k = (1 + x)^{-1} = \left(\sum_{k=0}^{\infty} c_k x^k\right)^m = \sum_{k=0}^{\infty}\sum c_{i_1}\cdots c_{i_m} x^k, \tag{2}$$

where the inner sum extends over all m-tuples $(i_1, \dots, i_m)$ of nonnegative integers summing to k. Exactly m such m-tuples have k as an entry. Equating coefficients of x^k in (2) then yields

$$mc_k + \sum c_{i_1} \cdots c_{i_m} = (-1)^k, \tag{3}$$

where the sum extends over m-tuples $(i_1, \dots, i_m)$ with sum k and entries less than k.

Let $R_m = \bigcup_{i \ge 0} (1/m^i)\mathbb{Z}$. Note that R_m is the subring of $\mathbb{Q}$ consisting of all rational numbers whose denominators factor into primes dividing m. Also, $c_0 = 1$, so $c_0 \in R_m$. Since m is a unit of R_m, (3) yields $c_k \in R_m$ for all k, inductively. Thus (**a**′) follows.

(**b**′) View (1) and (2) above in the formal power series ring $R_m[[x]]$. We write $f(x) \equiv g(x)$ when $f(x) - g(x) = ph(x)$ for some power series $h(x) \in R_m[[x]]$. Since $f(x)^p \equiv f(x^p)$ for all $f(x) \in R_m[[x]]$, also $f(x)^{m+1} \equiv f(x^{m+1})$. Therefore,

$$\sum_{k=0}^{\infty} c_k x^k = (1+x)^{-1/m} = (1+x)((1+x)^{-1/m})^{m+1} = (1+x)\left(\sum_{k=0}^{\infty} c_k x^k\right)^{m+1}$$

$$\equiv (1+x)\sum_{k=0}^{\infty} c_k x^{(m+1)k} = \sum_{k=0}^{\infty}(c_k x^{(m+1)k} + c_k x^{(m+1)k+1}). \tag{4}$$

We conclude that $c_k \equiv 0 \bmod p$ if k is not congruent to 0 or 1 modulo $m+1$, and the same holds for a_k.

Note that $c_0 = 1$ and $c_1 = -1/m \equiv 1 \bmod p$. Hence p divides neither a_0 nor a_1. For $k > 1$, if $m+1$ divides k or $k-1$, then write $k = (m+1)k' + \epsilon$, where $\epsilon \in \{0, 1\}$. Note that k is a sum of distinct powers of $m+1$ if and only if k' is. The congruence in (4) implies that $c_k \equiv c_{k'} \bmod p$, and (**b**′) follows by induction.

Also solved by R. Chapman (U. K.), H. Chen, P. Corn, P. P. Dályay (Hungary), Y. Dumont (France), E. Errthum, S. M. Gagola Jr., J. H. Lindsey II, O. P. Lossers (Netherlands), J. Minkus, M. A. Prasad (India), B. Schmuland (Canada), N. C. Singer, J. H. Smith, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

Convergence of a Prime-denominated Series

11384 [2008, 757]. *Proposed by Moubinool Omarjee, Lycée Jean-Lurçat, Paris, France.* Let p_n denote the nth prime. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{n} \rfloor}}{p_n}$$

converges.

Solution by Greg Martin, University of British Columbia, Vancouver, CA. Let $S_N = \sum_{n=1}^{N} \frac{(-1)^{\lfloor \sqrt{n} \rfloor}}{p_n}$. It suffices to show that the subsequence $\{S_{M^2-1} \colon M \ge 1\}$ converges, since S_N is between S_{M^2-1} and $S_{(M+1)^2-1}$ for N between M^2-1 and $(M+1)^2-1$. However, $S_{M^2-1} = \sum_{m=2}^{M} T_m$, where

$$T_m = \sum_{n=(m-1)^2}^{m^2-1} \frac{(-1)^{\lfloor \sqrt{n} \rfloor}}{p_n} = (-1)^{m-1} \sum_{n=(m-1)^2}^{m^2-1} \frac{1}{p_n}.$$

Since $\{T_m : m \geq 1\}$ alternates in sign, it suffices to show that $\lim T_m = 0$, by the alternating series test. Using the crude inequality $p_n > n$, we obtain

$$|T_m| < \sum_{n=(m-1)^2}^{m^2-1} \frac{1}{n} < \frac{1}{(m-1)^2} \sum_{n=(m-1)^2}^{m^2-1} 1 = \frac{2m-1}{(m-1)^2},$$

and thus $\lim T_m = 0$.

A similar proof works if the sequence of primes is replaced by an arbitrary sequence q satisfying $q_n/\sqrt{n} \to \infty$.

Editorial comment. Many solvers used detailed information about the distribution of the prime numbers, but the proof above shows that this is unnecessary.

Also solved by R. Bagby, H. Chen, P. P. Dályay (Hungary), Y. Dumont (France), V. V. Garcia (Spain), S. James (Canada), O. Kouba (Syria), K. Y. Li (China), J. Oelschlager, P. Perfetti (Italy), É. Pité (France), Á. Plaza (Spain), C. R. Pranesachar (India), M. T. Rassias (Greece), V. Schindler (Germany), B. Schmuland (Canada), N. C. Singer, A. Stadler (Switzerland), R. Stong, T. Tam, R. Tauraso (Italy) & M. Lerma, D. B. Tyler, J. Vinuesa (Spain), Z. Vörös (Hungary), GCHQ Problem Solving Group (U. K.), and the proposer.

Capturing Eigenvalues in an Interval

11387 [2008, 758]. *Proposed by Oskar Maria Baksalary, Adam Mickiewicz University, Poznań, Poland, and Götz Trenkler, Technische Universität Dortmund, Dortmund, Germany.* Let $\mathcal{C}_{n,n}$ denote the set of $n \times n$ complex matrices. Determine the shortest interval $[a, b]$ such that if P and Q in $\mathcal{C}_{n,n}$ are nonzero orthogonal projectors, that is, Hermitian idempotent matrices, then all eigenvalues of $PQ + QP$ belong to $[a, b]$.

Solution I by O. P. Lossers, Eindhoven University of Technology, Einhoven, The Netherlands. The eigenvalues of $PQ + QP$ lie in $[-\frac{1}{4}, 2]$. The matrix $P + Q$ is Hermitian, and hence there is an orthonormal basis of its eigenvectors. The eigenvalues of $P + Q$ are real and in $[0, 2]$, since $|(P + Q)x| \leq |Px| + |Qx| \leq 2|x|$. The matrix $PQ + QP$ equals $(P + Q)^2 - (P + Q)$ and thus has the same eigenvectors as $P + Q$, with eigenvalues of the form $\lambda^2 - \lambda$ with $0 \leq \lambda \leq 2$. It follows that the eigenvalues of $PQ + QP$ lie in $[-1/4, 2]$.

The maximum is attained when P and Q both equal the identity matrix, while the minimum is attained for the projections on two lines intersecting at an angle of $\pi/3$.

Solution II by Fuzhen Zhang, Nova Southeastern University, Fort Lauderdale, FL. Since $|(PQ + QP)x| \leq |PQx| + |QPx| \leq 2|x|$ for all x, the eigenvalues are at most 2. For the lower bound, write $X \geq Y$ if X and Y are Hermitian and $X - Y$ is positive semidefinite. Note that

$$0 \leq \left(P + Q - \frac{1}{2}I\right)^2 = P^2 + Q^2 + \frac{1}{4}I + PQ + QP - P - Q = \frac{1}{4}I + PQ + QP.$$

It follows that $PQ + QP \geq -\frac{1}{4}I$, and therefore each eigenvalue of $PQ + QP$ is at least $-\frac{1}{4}$.

For the extreme cases, taking $P = Q = I$ gives the largest eigenvalue 2. Setting $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $Q = \frac{1}{4}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}$ yields $-\frac{1}{4}$ as an eigenvalue.

Editorial comment. The part $PQ + QP \geq -\frac{1}{4}I$ of this problem appeared in F. Zhang, *Linear Algebra: Challenging Problems for Students* (2nd ed.), Johns Hopkins University Press, Baltimore, 2009, p. 81.

Also solved by R. Chapman (U. K.), J. Freeman, J.-P. Grivaux (France), J. Hartman, E. A. Herman, O. Kouba (Syria), T. Laffey & H. Šmigoc (Ireland), J. H. Lindsey II, M. Omarjee (France), R. Stong, S. E. Thiel, N. Thornber, Szeged Problem Solving Group "Fejéntaláltuka" (Hungary), GCHQ Problem Solving Group (U. K.), and the proposers.

Distinct Multisets with the Same Pairwise Sums

11389 [2008, 758]. *Proposed by Elizabeth R. Chen and Jeffrey C. Lagarias, University of Michigan, Ann Arbor, MI.* Given a multiset $A = \{a_1, \ldots, a_n\}$ of n real numbers (not necessarily distinct), define the sumset $S(A)$ of A to be $\{a_i + a_j : 1 \le i < j \le n\}$, a multiset with $n(n-1)/2$ not necessarily distinct elements. For instance, if $A = \{1, 1, 2, 3\}$, then $S(A) = \{2, 3, 3, 4, 4, 5\}$.

(a) When n is a power of 2 with $n \ge 2$, show that there are two distinct multisets A_1 and A_2 such that $S(A_1) = S(A_2)$.

(b) When n is a power of 2 with $n \ge 4$, show that if r distinct multisets $A_1, \ldots, A_r$ all have the same sumset, then $r \le n - 2$.

(c*) When n is a power of 2 with $n \ge 4$, can there be as many as 3 distinct multisets with the same sumset?

(Distinct multisets are known to have distinct sumsets when n is not a power of 2.)

Solution by BSI Problems Group, Bonn, Germany.

(a) We recursively construct multisets A_m and B_m of size 2^m for $m \ge 0$. For $m \ge 0$, choose arbitrary positive c_m. Let $A_0 = \{0\}$ and $B_0 = \{c_0\}$. For $m > 0$, let $A_m = A_{m-1} \cup \{b + c_m : b \in B_{m-1}\}$ and $B_m = B_{m-1} \cup \{a + c_m : a \in A_{m-1}\}$. Inductively, $|A_m| = |B_m| = 2^m$ and $S(A_m) = S(B_m)$. Also $\min A_m = 0 < \min B_m$, which yields $A_m \ne B_m$.

(b) First we prove three claims. Let $A = \{a_1, \ldots, a_n\}$ with $a_1 \le \cdots \le a_n$, and let $S(A) = \{s_1, \ldots, s_{n(n-1)/2}\}$ with $s_1 \le \cdots \le s_{n(n-1)/2}$.

Claim 1: $a_2 + a_3 \in \{s_3, \ldots, s_n\}$. Since $a_1 + a_2 \le a_1 + a_3 \le a_2 + a_3$, we have $a_2 + a_3 \ge s_3$. Also, the only sums that can be strictly smaller than $a_2 + a_3$ are $\{a_1 + a_i : 2 \le i \le n\}$. Thus $a_2 + a_3 \le s_n$.

Claim 2: Let $B = \{b_1, \ldots, b_n\}$ with $b_1 \le \cdots \le b_n$. If $a_1 = b_1$ and $S(A) = S(B)$, then $A = B$. We prove $a_i = b_i$ by induction on i. Let $A(i) = \{a_1, \ldots, a_i\}$ and $B(i) = \{b_1, \ldots, b_i\}$. If $A(i-1) = B(i-1)$, then $a_1 + a_i$ and $b_1 + b_i$ are both minimal among $S(A) - S(A(i-1))$. Thus $a_{i+1} = b_{i+1}$.

Claim 3: Let $B = \{b_1, \ldots, b_n\}$ with $b_1 \le \cdots \le b_n$. If $a_2 + a_3 = b_2 + b_3$ and $S(A) = S(B)$, then $A = B$. Since the two smallest sums from the two sets are equal, $a_1 + a_2 = s_1 = b_1 + b_2$ and $a_1 + a_3 = s_2 = b_1 + b_3$. With the hypothesis $a_2 + a_3 = b_2 + b_3$, we have $a_1 = b_1$. Claim 2 now applies.

Given these claims, let $A^1, \ldots, A^{n-1}$ be multisets of size n having the same sumset. Write $A^k = \{a_1^{(k)}, \ldots, a_n^{(k)}\}$ with $a_1^{(k)} \le \cdots \le a_n^{(k)}$. By Claim 1, there are at most $n - 2$ values for the sum of the second and third smallest elements. By the pigeonhole principle, there exist distinct k and l such that $a_2^{(k)} + a_3^{(k)} = a_2^{(l)} + a_3^{(l)}$. By Claim 3, $A_k = A_l$. Thus at most $n - 2$ multisets can have the same sumset.

(c) The answer is yes. Let $A = \{0, 4, 4, 4, 6, 6, 6, 10\}$, $B = \{2, 2, 2, 4, 6, 8, 8, 8\}$, and $C = \{1, 3, 3, 3, 7, 7, 7, 9\}$. With exponents denoting multiplicity, $S(A)$, $S(B)$, and $S(C)$ all equal $\{4^{(3)}, 6^{(3)}, 8^{(3)}, 10^{(10)}, 12^{(3)}, 14^{(3)}, 16^{(3)}\}$.

Editorial comment. The GCHQ Problem Solving Group solved part (a) by letting A_1 be the set of nonnegative integers less than $2n$ whose binary expansion has an even number of ones and setting $A_2 = \{0, 1, \ldots, 2n - 1\} - A_1$. This results from the construction given above by setting $c_m = 2^m$.

For part (c), Daniele Degiorgi gave the example $A = \{0, 6, 7, 9, 11, 13, 14, 20\}$, $B = \{1, 5, 6, 8, 12, 14, 15, 19\}$, and $C = \{2, 4, 5, 9, 11, 15, 16, 18\}$, showing that it can be solved with sets (i.e., multisets with no repeated elements).

It remains open whether there are quadruples of multisets of size greater than 2 with the same sumset, or whether there are triples of multisets of any size greater than 2 other than 8 with the same sumset. Richard Stong showed that the search for such triples can be restricted to multisets whose size is an odd power of 2.

Also solved by D. Degiorgi (Switzerland), R. Stong, and the GCHQ Problem Solving Group (U. K.). Parts (a) and (b) solved also by O. P. Lossers (Netherlands), M. A. Prasad (India), Microsoft Research Problems Group, and the proposers.

Tetrahedral Cevians

11405 [2009, 82]. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.* Let P be an interior point of a tetrahedron $ABCD$. When X is a vertex, let X' be the intersection of the opposite face with the line through X and P. Let XP denote the length of the line segment from X to P.
(**a**) Show that $PA \cdot PB \cdot PC \cdot PD \geq 81 PA' \cdot PB' \cdot PC' \cdot PD'$, with equality if and only if P is the centroid of $ABCD$.
(**b**) When X is a vertex, let X'' be the foot of the perpendicular from P to the plane of the face opposite X. Show that $PA \cdot PB \cdot PC \cdot PD = 81 PA'' \cdot PB'' \cdot PC'' \cdot PD''$ if and only if the tetrahedron is regular and P is its centroid.

Solution by Kit Hanes, Bellingham, WA. We will consider the more general case of an n-simplex with vertices $A_0, \ldots, A_n$. Let P be a point in the interior, and let A_i' be the point where the line $A_i P$ meets the face opposite A_i. We will show that $\prod_{k=0}^{n} PA_k \geq n^{n+1} \prod_{k=0}^{n} PA_k'$, with equality if and only if P is the centroid of the simplex. Let $P = a_0 A_0 + \cdots + a_n A_n$ where $a_0 + \cdots + a_n = 1$ and each a_i is positive. For each j, A_j' is a convex combination of the A_i with A_j omitted and P is a convex linear combination of A_j and A_j'. Hence $P = a_j A_j + (1 - a_j) A_j'$. Hence $PA_j / PA_j' = (1 - a_j)/a_j$. The inequality of (**a**) is equivalent to $\prod_{j=0}^{n} (1 - a_j) \geq n^{n+1} \prod_{j=0}^{n} a_j$. This inequality follows by applying the arithmetic-geometric mean inequality

$$\frac{1 - a_j}{n} = \frac{a_0 + \cdots + \widehat{a_j} + \cdots + a_n}{n} \geq \sqrt[n]{a_0 \ldots \widehat{a_j} \ldots a_n}$$

to each term separately and taking the product. (Here, the hats indicate that the hatted term is to be skipped.) Equality holds if and only if all the a_i are equal, and hence $a_i = 1/(n+1)$ for all i and P is the centroid of the simplex. For part (**b**), note that $PA_i' \geq PA_i''$ with equality if and only if $A_i' = A_i''$, i.e., if and only if the line PA_i is an altitude of the simplex. Hence the stated equality holds exactly when P is both the centroid and the orthocenter of the simplex. That this is equivalent to the simplex being regular is half of Problem 11087 from this MONTHLY, December, 2005.

Editorial comment. Part (a) of this problem is the generalization from triangles to tetrahedra of Problem 11325, this MONTHLY, November, 2007.

Also solved by S. Amghibech (Canada), M. Bataille (France), M. Can, R. Chapman (U. K.), P. P. Dályay (Hungary), O. Geupel (Germany), M. Goldenberg & M. Kaplan, J. Grivaux (France), K. Hanes, J. G. Heuver (Canada), B.-T. Iordache (Romania), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), J. Schaer (Canada), R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.