Calculus I Name: ______________________________

Fall 2007

Instructions: You have 65 minutes for this exam. The exam is closed book, closed notes. A calculator is allowed, but you must show all of your work. **Your work is your answer.** If you have any questions, please ask immediately! Good luck.

PRACTICE EXAM

Problem #1:

Find the derivative of the following (A is a constant):

a) $f(x) = \frac{A^2}{x^7} - A\sqrt[16]{x - A}$

Answer:

$$f'(x) = -7A^2x^{-8} - (A/16)(x - A)^{-15/16}$$

b) $f(x) = x^e - Ae^x + \ln(A^5x) - \sin(Ax)$

Answer:

$$f'(x) = ex^{e-1} - Ae^x + x^{-1} - A\cos(Ax)$$

Problem #2:

Find all antiderivatives of $g(t) = t^{1/5} - \frac{1}{\sqrt{2}t^3} + \cos(t)$

Answer:

$$G(t) = (5/6)t^{6/5} + \frac{1}{2t^2\sqrt{2}} + \sin(t) + C$$

Problem #3:

Check that $y = -\frac{1}{t+C}$ is a solution to the differential equation:

$$y' = y^2.$$

Answer:

Calculate $y' = (t + C)^{-2}$ and see that this equals:

$$1/y^2 = 1/(t + C)^2.$$

Problem #4:

Let $h(x) = e^x - \ln(x)$.

a) Find the equation of the line throught the points $(3-d, h(3-d))$ and $(3, h(3))$. **Answer:** The

slope of the tangent line is:

$$m = \frac{e^{3-d} - \ln(3-d) - e^3 + \ln 3}{-d}.$$

The equation of the line with this slope through $(3, h(3))$ is:

$$y = \Big(\frac{e^{3-d} - \ln(3-d) - e^3 + \ln 3}{-d}\Big)(x-3) + e^3 - \ln 3.$$

b) Carefully explain how you could use your answer from part a) to calculate $h'(3)$. Your answer should use a diagram to help your explanation.

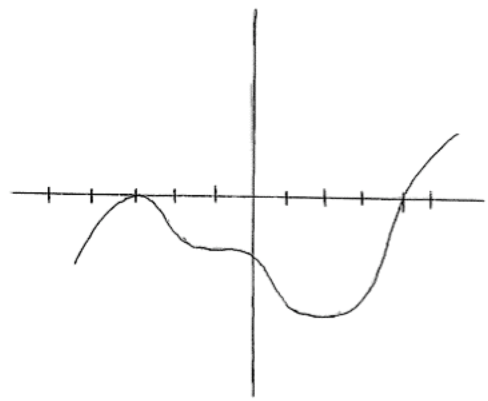
Partial Answer: As $d \to 0$, the slope of the line in part a) approaches the derivative of $h(x)$ at $x = 3$. So to calculate $h'(3)$ just plug smaller and smaller values of d into the slope from part a).

c) Find the equation of the line tangent to $h(x)$ at $x = e$. (You should not use your work from parts a) and b).)

Answer: $h'(x) = e^3 - (1/x)$ so $h'(e) = e^e + (1/e)$. The equation of the tangent line is then

$$y = (e^e + \frac{1}{e})(x - e) + (e^3 - 1).$$

For Problems #5 and #6 use the graph of the function $f(x)$ below.



Problem #5:

Sketch (on a new set of axes) the graph of the **derivative** for $f(x)$.

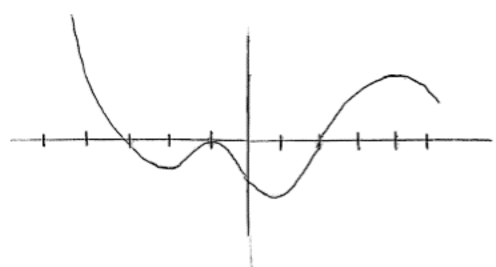


FIGURE 1. The derivative of $f(x)$.

Problem #6:

Sketch (on a new set of axes) the graph of a possible antiderivative for $f(x)$.

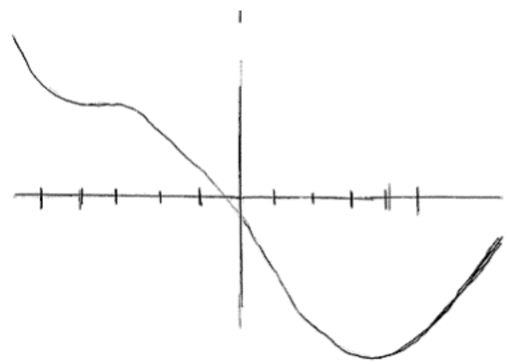


FIGURE 2. A possible antiderivative for $f(x)$.

Problem #7:

Consider $f(x) = \frac{1}{\sqrt{x-2}}$. Find $f'(x)$ **using the definition of the derivative**. No credit will be given for any other method of obtaining the answer.

Answer:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h-2}} - \frac{1}{\sqrt{x-2}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x-2}-\sqrt{x+h-2}}{h\sqrt{x-2}\sqrt{x+h-2}} \cdot \frac{\sqrt{x-2}+\sqrt{x+h-2}}{\sqrt{x-2}+\sqrt{x+h-2}} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{x-2}\sqrt{x+h-2}(\sqrt{x-2}+\sqrt{x+h-2})} \\
&= \frac{-1}{2(x-2)\sqrt{x-2}}
\end{aligned}
$$

Problem #8:

Find the point on the graph of $y = \frac{1}{x} + x$ which is closest to the origin.

(Hint: the distance between two points is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, but you may minimize the **square** of the distance.)

Answer: Let $s(x)$ be the square of the distance from the point $(x, f(x))$ to $(0, 0)$. Then

$$s(x) = x^2 + (x^{-1} + x)^2.$$

Rewrite this as:

$$s(x) = 2x^2 + 2 + x^{-2}.$$

We wish to find out where this achieves its maximum. Take the derivative:

$$s'(x) = 4x - \frac{2}{x^3}.$$

Set equal to zero:

$$4x - \frac{2}{x^3} = 0.$$

Multiply by x^3:

$$4x^4 - 2 = 0.$$

Solve for x:

$$x = \pm\frac{1}{\sqrt[4]{2}}.$$

We should verify that these are both maxima:

$$s''(x) = 4 + \frac{6}{x^4}.$$

This function is always positive and so $s(x)$ always concave up and so both our stationary points: $x = \pm\frac{1}{\sqrt[4]{2}}$ are minima.

Problem #9:

Gilda Smith is a goldsmith in Gaul who gilds goblets. She is going to gild the outside of a triangular goblet with volume 200 cm^3. The ends of the goblet are equilateral triangles with sides of length s cm (one of the ends is open). The goblet is w cm tall. She is galled to discover that the goldsmiths' guild specifies that the gold for the triangular bottom of the goblet costs 6 guilders per cm^2 and the gold for the sides costs 10 guilders per cm^2. What length s will minimize the total cost for the gold?

The area of an equilateral triangle with sides of length s is $\frac{\sqrt{3}}{2}s^2$. The volume of the goblet described is $\frac{\sqrt{3}}{2}s^2 w$.

Answer: We are told that:

$$\frac{\sqrt{3}}{2}s^2 w = 200.$$

The cost of gilding the goblet is:

$$C = 3\sqrt{3}s^2 + 30ws.$$

Use the formula for volume to rewrite this as:

$$C(s) = 3\sqrt{3}s^2 + \frac{12000}{\sqrt{3}s}.$$

Take the derivative:

$$C'(s) = 6\sqrt{3}s - \frac{12000}{\sqrt{3}s^2}.$$

Set equal to zero to find stationary points of $C(s)$:

$$6\sqrt{3}s - \frac{12000}{\sqrt{3}s^2} = 0$$

Solve:

$$s = \sqrt[3]{\frac{12000}{18}}.$$

To verify that this is the **minimum** cost, take the second derivative:

$$C''(s) = 6\sqrt{3} + \frac{24000}{\sqrt{3}s^3}$$

This is always positive, since $s > 0$. Hence, $C(s)$ is always concave up and so the value for s we found is the minimum of the function.