The Fundamental Theorem of Calculus

Theorem (FTC 1)**.** *Suppose that $f(x)$ is a continuous function on an open interval containing $a \in \mathbb{R}$. Then:*

$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$$

Proof. By the definition of the derivative:

$$\frac{d}{dx}\int_a^x f(t)\ dt = \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt\right]$$

We wish to show that this is equal to $f(x)$. We will in fact show this only for:

$$\frac{d}{dx}\int_a^x f(t)\ dt = \lim_{h\to 0^+}\frac{1}{h}\left[\int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt\right]$$

That is, we are assuming that $h > 0$. The case when $h < 0$ is similar.

We then have,

$$\lim_{h\to 0^+}\frac{1}{h}\left[\int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt\right] = \lim_{h\to 0^+}\frac{1}{h}\int_x^{x+h} f(t)\ dt$$

An explanation for this fact can be seen in Figure 1. It also follows from the basic properties of integrals:

$$\int_0^{x+h} f(t)\ dt = \int_0^x f(t)\ dt + \int_x^{x+h} f(t)\ dt.$$

We now examine the integral $\int_x^{x+h} f(t)\ dt$.

The function $f(t)$ is continuous (by hypothesis) on the interval $[x, x+h]$. By the extreme value theorem it achieves a maximum value $M(h)$ and a minimum value $m(h)$ on this interval. In particular, for $t \in [x, x+h]$

$$m(h) \leq f(t) \leq M(h)$$

Since $x + h > x$, taking definite integrals preserves inequalities, so:

$$\int_x^{x+h} m(h)\ dt \leq \int_x^{x+h} f(t)\ dt \leq \int_x^{x+h} M(h)\ dt$$

The leftmost integral represents the area of a rectangle with base $(x + h) - x$ and height $m(h)$. The rightmost integral represents the area of

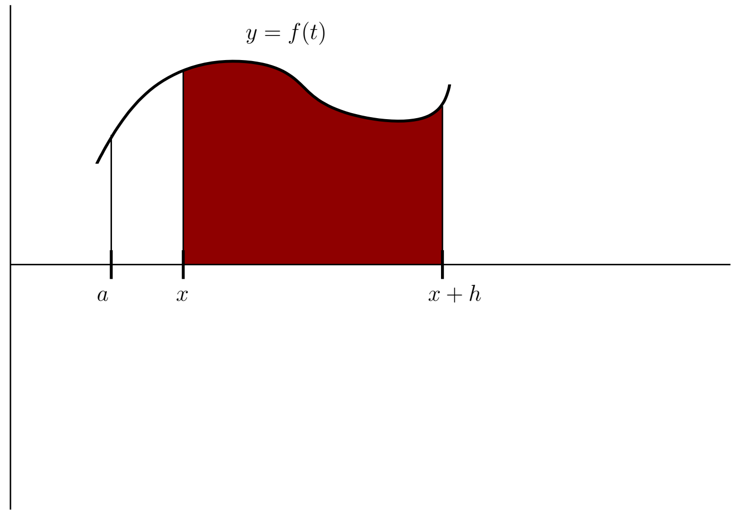

FIGURE 1. The integral $\int_x^{x+h} f(t)\ dt$ is represented by the shaded area.

a rectangle with base $(x + h) - x$ and height $M(h)$. See Figure **??**. Thus:

$$hm(h) \leq \int_x^{x+h} f(t)\ dt \leq hM(h).$$

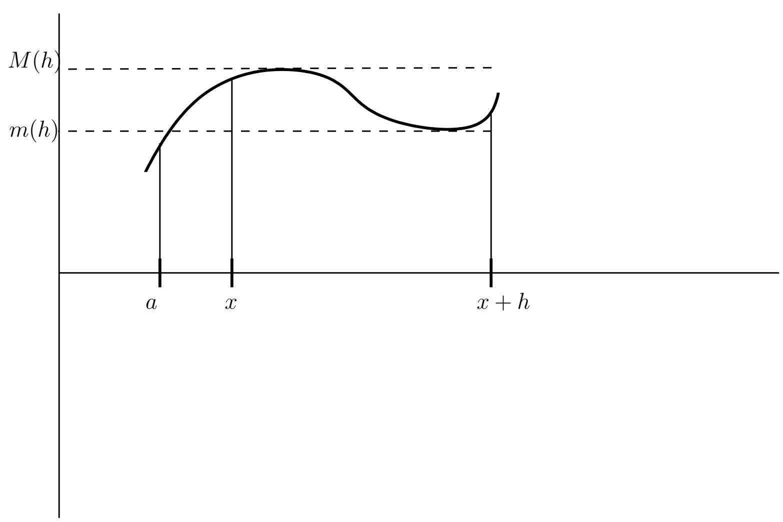

FIGURE 2. Can you find the rectangles discussed above?

Since $h > 0$ dividing by it doesn't change the direction of the inequalities:

$$m(h) \leq \frac{1}{h} \int_x^{x+h} f(t)\ dt \leq M(h)$$

Now as $h \to 0$, $m(h) \to f(x)$ and $M(h) \to f(x)$ since both are in between $f(x)$ and $f(x+h)$ for all values of h. Taking limits preserves inequalities so we have:

$$\lim_{h \to 0} m(h) \leq \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\ dt \leq \lim_{h \to 0} M(h)$$

Which means:

$$f(x) \leq \lim_{h \to 0} \int_x^{x+h} f(t)\ dt \leq f(x)$$

By the squeeze theorem,

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\ dt = f(x)$$

In other words,

$$\frac{d}{dx} \int_a^x f(t)\ dt = f(x)$$

□

Theorem (FTC 2)**.** *Suppose that $f(x)$ is a continuous function on an open interval containing the closed interval $[a, b]$ and that $F(x)$ is any antiderivative of $f(x)$. Then*

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Proof. By FTC 1, the function $G(x) = \int_a^x f(t)\ dt$ is an antiderivative for $f(x)$. By a corollary to the Mean Value Theorem, $F(x) = G(x) + C$ where C is some constant. Thus,

$$F(b) - F(a) = \Big(G(b) + C\Big) - \Big(G(a) + C\Big) \tag{1}$$

Now, $G(a) = \int_a^a f(t)\ dt$. This equals 0.

Also, $G(b) = \int_a^b f(t)\ dt$. Plugging these facts into Equation 1, gives:

$$F(b) - F(a) = \int_a^b f(t)\ dt.$$

□