**Calc 1: Practice Exam 2 Solutions**

**Name:**

(1) Find derivatives of the following functions:

(a)

$$y = \frac{x^5 + 2x + 1}{\sqrt[3]{x}\cos(x)}$$

**Answer:** Use the quotient and product rules:

$$y' = \frac{\sqrt[3]{x} cos(x)(5x^4 + 2) - (x^5 + 2x + 1)(\frac{1}{3}x^{-2/3}\cos(x) - x^{1/3}\sin(x))}{(\sqrt[3]{x}\cos(x))^2}$$

(b)

$$y = e^{x^3}\tan(x^4 + 1)$$

**Answer:** Use the product rule and the chain rule:

$$y' = 3x^2(e^{x^3})\tan(x^4 + 1) + e^{x^3}\sec^2(x^4 + 1)(4x^3)$$

(c)

$$y = \cos(e^{x^2})\sin(e^{-x^2})$$

**Answer:** Use the product rule and the chain rule:

$$y' = -\sin(e^{-x^2})\sin(e^{x^2})(e^{x^2})(2x) + \cos(e^{x^2})\cos(e^{-x^2})(e^{-x^2})(-2x)$$

(d)

$$y = \frac{\tan(\sqrt{x} - \sqrt[3]{(x)})}{x^2e^{5x}}$$

**Answer:** Use the quotient rule, the product rule, and the chain rule.

$$y' = \frac{x^2e^{5x}\sec^2(\sqrt{x}-\sqrt[3]{x})(\frac{1}{2}x^{-1/2}-\frac{1}{3}x^{-2/3}) - \tan(\sqrt{x}-\sqrt[3]{x})(2xe^{5x}+5x^2e^{5x})}{x^4e^{10x}}$$

(2) Consider the curve defined by the equation

$$ye^x + xe^y = e^{xy}$$

(a) Find $\frac{dy}{dx}$

**Answer:** Take the derivative with respect to $x$ of both sides, using implicit differentiation:

$$ye^x + e^x\frac{dy}{dx} + e^y + xe^y\frac{dy}{dx} = e^{xy}(x\frac{dy}{dx} + y)$$

Solve for $\frac{dy}{dx}$ to find:

$$\frac{dy}{dx} = \frac{ye^{xy} - ye^x - e^y}{e^x + xe^y - xe^{xy}}$$

(b) Find the equation of the tangent line to the curve at the point (1,0).

**Answer:** In the expression above, plug in $x = 1$ and $y = 0$ to find that the slope of the tangent line is: $\frac{-1}{e}$. Thus the equation of the tangent line is:

$$y = -\frac{1}{e}(x-1)$$

(3) Show all the steps for finding the derivative of:

(a) $y = \arcsin(x)$

(b) $y = \arccos(x)$

(c) $y = \arctan(x)$

**Answer:** Rewrite $y = \arctan(x)$ as $x = \tan(y)$. Differentiate both sides with respect to $x$ to find that: $1 = \sec^2(y)\frac{dy}{dx}$. Thus:

$$\frac{dy}{dx} = \frac{1}{\sec^2(\arctan(x))}$$

To simplify this make a right triangle which has an angle $y$ (since $\arctan(x)$ spits out the angle $y$). Since tangent is opposite over adjacent, label the side opposite the angle $y$ with $x$ and the side adjacent with a 1. Thy hypotenuse then has length $\sqrt{1+x^2}$ by Pythagoras' theorem. The function secant is defined as hypotenuse over adjacent, so $\sec(\arctan(x)) = \sec(y) = \sqrt{1+x^2}$. Thus:

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

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(4) Suppose that $f(t)$ is a solution of the DE $y' = t^2y + t$ and that $f(1) = 2$. Find the equation of the tangent line to $f(t)$ at the point (1,2).

**Answer:** The slope of the tangent line is $f'(t)$ when $t = 1$ and $y = 2$. The fact that $f(t)$ is a solution to the given DE, tells us that $f'(t) = t^2f(t) + t$. So when $t = 1$ and $y = 2$ we have: $f'(1) = (1)^2(2) + 1$. In other words, the slope is 3. Thus the equation of the tangent line is $y = 3(t-1) + 2$.

(5) Suppose that $h(t)$ is a solution of the DE $y' = e^ty$ and that $h(-2) = 1$. Find an equation of the line tangent to $h$ at the point $(-2, 1)$.

**Answer:** This works the same way as the previous one. The slope is $h'$ when $t = -2$ and $h = 1$. Thus the slope is $e^{-2}(1) = e^{-2}$. The equation of the tangent line to $h(t)$ at the point $(-2, 1)$ is:

$$y = e^{-2}(t+2) + 1$$

(6) Find the following limits. Explain all your steps.

(a)

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}}$$

**Answer:** The given limit has the indeterminate form $\frac{\infty}{\infty}$ so we can use l'Hopital's rule. We find that the given limit is the same as:

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}}$$

We can rewrite this as:

$$\lim_{x\to\infty} \frac{3}{x^{1/3}}$$

Since the denominator goes to $+\infty$ and the numerator stays at 3 as $x$ goes to $\infty$ the given limit evaluates to 0.

(b)

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$$

**Answer:** The given limit has the indeterminate form $\frac{0}{0}$ so we can use l'Hopital's rule to see that the given limit is equal to:

$$\lim_{x\to 0} \frac{e^x - 1}{2x}$$

This limit still has the indeterminate form $\frac{0}{0}$ so we apply l'Hopital's rule again:

$$\lim_{x\to 0} \frac{e^x}{2}$$

As $x \to 0$ the function $e^x$ heads toward 1 and 2 stays at 2, so the overall limit is equal to $\frac{1}{2}$.

(c)

$$\lim_{x\to\infty} \frac{x^2 - 1}{2x^2 + 1}$$

**Answer:** We could use l'Hopital's rule here, but it is simpler to multiply both numerator and denominator of the fraction by $\frac{1}{x^2}$ converting it into the limit:

$$\lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}}$$

The function $\frac{1}{x^2} \to 0$ as $x \to \infty$ so the overall limit is $\frac{1}{2}$.

(7) Find the point on the line $y = -3x + 2$ which is closest to the origin.

**Answer:** Define the function $d(x) = \sqrt{x^2 + (-3x+2)^2}$ which represents the distance of the point $(x, y)$ (where $y = -3x + 2$) from the origin. We wish to minimize this so we take the derivative using the chain rule:

$$d'(x) = \frac{2x + 2(-3x+2)(-3)}{2\sqrt{x^2 + (-3x+2)^2}}$$

Note that the denominator is 2 times $d(x)$ so since the line doesn't pass through the origin, the denominator is never zero. Thus the only critical points occur when the numerator is zero. To see when this occurs, solve the following:

$$2x + 2(-3x+2)(-3) = 0$$

We see that $x = \frac{3}{5}$. So the point nearest the origin is $(\frac{3}{5}, \frac{1}{5})$.

(8) When a person coughs, the velocity of the air stream is related to the radius of the trachea by:

$$v(r) = k(r_0 - r)r^2$$

The normal radius of the trachea is $r_0$ and $k$ is a constant. The radius $r$ of the trachea is restricted so that $\frac{1}{2}r_0 \leq r \leq r_0$ in order to prevent suffocation.

Find the value of $r$ which makes $v$ an absolute maximum.

**Answer:** Rewrite: $v(r) = kr_0r^2 - kr^3$. Then differentiate:

$$v'(r) = 2kr_0r - 3kr^2$$

The derivative always exists, so the only critical points are where $v'(r) = 0$:

$$\begin{array}{ccc} 2kr_0r - 3kr^2 & = & 0 & \Rightarrow \\ r(2kr_0 - 3kr) & = & = & \Rightarrow \\ r = 0 & \text{or} & r = \frac{2r_0}{3} \end{array}$$

Now, when $r = 0$ the person has suffocated, so we need only evaluate $v(r)$ when $r = \frac{2r_0}{3}$ and when $r$ equals one of the end-points:

$$\begin{array}{ccccc} v(\frac{2r_0}{3}) & = & \frac{4kr_0^3}{9} - \frac{8r_0^3}{27} & = & \frac{(12k+8)r_0^3}{27} \\ v(\frac{1}{2}r_0) & = & \frac{kr_0^3}{4} - \frac{kr_0^3}{8} & = & \frac{kr_0^3}{4} \\ v(r_0) & = & kr_0^3 - kr_0^3 & = & 0 \end{array}$$

Since $\frac{12k}{27} > \frac{k}{4}$ it is straightforward to see that $\frac{(12k+8)r_0^3}{27} > \frac{kr_0^3}{4}$. Thus, $r = \frac{2r_0}{3}$ gives the maximum velocity.

(9) Find the speed of the following curve at the point when $t = 0$:

$$\begin{cases} x(t) & = & 2\sin(4t) \\ y(t) & = & 3\sin(5t) \end{cases}$$

**Answer:** The speed is defined to be $\sqrt{(x')^2 + (y')^2}$. Differentiating $x$ and $y$ we obtain: $x'(t) = 8\cos(4t)$ and $y'(t) = 15\cos(5t)$. When $t = 0$ we have $x'(0) = 8$ and $y'(0) = 15$. So the speed when $t = 0$ is $\sqrt{8^2 + 15^2}$.

(10) If two resistors with resistances $R_1$ and $R_2$ are placed in parallel in a circuit, the total resistance of the circuit, $R$ is related to $R_1$ and $R_2$ be the equation:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If $R_1$ is increasing at a rate of .4 ohms per second and $R_2$ is increasing at a rate of .6 ohms per second, how fast is $R$ changing when $R_1 = 20$ ohms and $R_2 = 30$ ohms?

**Answer:** We know that $\frac{dR_1}{dt} = .4$ and $\frac{dR_2}{dt} = .6$. We wish to find $\frac{dR}{dt}$. Differentiating the given equation with respect to $t$ we obtain:

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt}$$

Plugging in the values for $\frac{dR_1}{dt}$ and $\frac{dR_2}{dt}$ and dividing the whole thing by $(-1)$ we obtain:

$$\frac{1}{R^2}\frac{dR}{dt} = \frac{1}{R_1^2}(.4) + \frac{1}{R_2^2}(.6)$$

When $R_1 = 20$ and $R_2 = 30$ we solve and find out that $R = 12$. Thus:

$$\frac{1}{144}\frac{dR}{dt} = \frac{1}{400}(.4) + \frac{1}{900}(.6)$$

So:

$$\frac{dR}{dt} = (\frac{1}{1000} + \frac{2}{3000})144 = .24$$

(11) A baseball diamond is a square with sides 90 ft. A batter hits the ball and runs toward first base with a speed of 24 feet per second. At what rate is his distance from 3rd base increasing, when he is halfway to 1st base? (The bases are located at the corners of the square).

**Answer:** Draw a picture. The path from 1st base to 3rd base (straight across the field) to home is a right triangle. When the player is $x$ feet away from home the distance to 3rd base is $d(x) = \sqrt{x^2 + 90^2}$. Rewrite this as:

$$d^2 = x^2 + 90^2$$

Taking the derivative with respect to t:

$$2d\frac{dd}{dt} = 2x\frac{dx}{dt}$$

Thus:

$$\frac{dd}{dt} = \frac{x}{d}\frac{dx}{dt}$$

We know that $\frac{dx}{dt} = 24$. When $x = 45$ we have $d = 45\sqrt{5}$. Thus when the player is halfway to first base:

$$\frac{dd}{dt} = \frac{24}{\sqrt{5}}$$

(12) Explain why the function $y = 2000x^{91} - 24x^2 + 15$ must have at least one root. (Your answer should involve a theorem discussed in class.)

**Answer:** When $x = 1$ we can see that $y > 0$ and when $x = -1$, $y < 0$. Since $y$ is a continuous function, the intermediate value theorem tells us that we must have some number $c$ between -1 and 1 such that $f(c) = 0$. This is a root of the function.

(13) When I wake up in the morning it is 45° F. At lunch time it is 68° F. Explain how I know that there was some time in the morning when it was exactly $15\pi$°F.

**Answer:** Let $f(t)$ be the temperature at time $t$. Temperature is a continuous function of time. When $t$ represents a time early in the morning $f(t) = 45$, when $t$ represents lunchtime $f(t) = 68$. The number $15\pi$ is between 45 and 68 so the intermediate value theorem tells me that there must be sometime in the morning when the temperature was exactly $15\pi$ degrees.

(14) Let $f(t) = t^3 + t + 3$. Explain why there is some number $c$ between -1 and 1 such that $f'(c) = 2$. Your answer should use a theorem we discussed in class.

**Answer:** The average value of the function $f$ on the interval $[-1, 1]$ is:

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{5 - (1)}{2} = 2$$

Since $f(t)$ is both continuous and differentiable we can apply the mean value theorem which says that there must be some number $c$ between $-1$ and 1 at which our function achieves its average value on the interval.