Math 10: Practice Final Answer Key

Name:

The answers below are not comprehensive and are meant to indicate the correct way to solve the problem.

Problem 1: Consider the definite integral

$$I = \int_1^5 \sin\left(\frac{1}{x}\right) dx.$$

(1) Using 4 subintervals, find the left hand approximation L_4 to I.

Answer: $\sin(1) + \sin(1/2) + \sin(1/3) + \sin(1/4)$

(2) Using 4 subintervals, find the right hand approximation R_4 to I.

Answer: $\sin(1/2) + \sin(1/3) + \sin(1/4) + \sin(1/5)$

(3) Using 4 subintervals, find the midpoint approximation M_4 to I.

Answer: $\sin(2/3) + \sin(2/5) + \sin(2/7) + \sin(2/9)$

(4) Using 4 subintervals, find the trapezoidal approximation T_4 to I.

Answer: $(1/2)\sin(1) + \sin(1/2) + \sin(1/3) + \sin(1/4) + (1/2)\sin(1/5)$

(5) Using the error bounds discussed in class, estimate $|I - R_4|$.

Answer:

$$|f'(x)| = |(-1/x^2)\cos(1/x)| \leq 1/x^2$$

Since $x \in [1,5]$ and since $1/x^2$ is decreasing $|f'(x)| \leq 1$ for all $x \in [1,5]$. Choose $K_1 = 1$. Then:

$$|I - R_4| \leq 2.$$

(6) Using the error bounds discussed in class, estimate $|I - M_4|$.

Answer:

$$|f''(x)| = \left|(1/x^2)(-1/x^2)\sin(1/x) + (2/x^3)\cos(1/x)\right| \leq \left|(-1/x^4)\right| + \left|2/x^3\right| \leq 3$$

when $x \in [1,5]$. Choose $K_2 = 3$. Then:

$$\left| I - M_4 \right| \leq 3 \cdot 4^3/(24 \cdot 16)$$

Problem 2: Consider the initial value problem

$$\begin{gathered} y' = y - t \\ y(0) = 0 \end{gathered}$$

Using 4 time steps of size $\Delta t = \frac{1}{4}$, estimate $y(1)$.

Answer:

$$\begin{aligned}
y(0) &= 0 \\
y(1/4) &\approx y(0) + f(0, y(0)) \cdot (1/4) = 0 \\
y(1/2) &\approx y(1/4) + f(1/4, y(1/4)) \cdot (1/4) = -(1/16) \\
y(3/4) &\approx y(1/2) + f(1/2, y(1/2)) \cdot (1/4) = -(1/16) - (5/64) \\
y(1) &\approx y(3/4) + f(3/4, y(3/4)) \cdot (1/4) = [-(1/16) - (5/64)] + [(-1/16) - (5/64) - (3/4)]
\end{aligned}$$

Problem 3: Find the arc length of the following curves:

(1) $f(x) = \ln(x)$ from $x = 1/\sqrt{3}$ to $x = \sqrt{3}$

Answer:

$$\sqrt{1 + f'(x)^2} = \frac{\sqrt{x^2+1}}{x}$$

So the arc length is

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{\sqrt{x^2+1}}{x}\, dx$$

Perform trig substitution with $x = \tan\theta$ to get:

$$\int_{\pi/6}^{\pi/3} \frac{\sec^3\theta}{\tan\theta}\, d\theta$$

Rewrite as:

$$\int_{\pi/6}^{\pi/3} \frac{(\tan^2\theta + 1)\sec\theta}{\tan\theta}\, d\theta$$

Which equals

$$\int_{\pi/6}^{\pi/3} \tan\theta\sec\theta + \csc\theta\, d\theta$$

Finding antiderivatives we get:

$$\sec\theta + \ln|\csc\theta - \cot\theta|\Big|_{\pi/6}^{\pi/3}$$

Plug in and obtain:

$$(2/\sqrt{3}) + \ln(2-\sqrt{3}) - (2 + \ln(1/\sqrt{3}))$$

(2) $f(x) = x^2$ from $x = 0$ to $x = 1$

Answer:

$$\sqrt{1 + (f'(x))^2} = \sqrt{1 + 4x^2}$$

The arc length is then $\int_0^1 \sqrt{1+4x^2}\, dx$. Use trig substitution with $x = (1/2)\tan\theta$ to obtain:

$$(1/2)\int_0^{\pi/4} \sqrt{1+\tan^2\theta}\,\sec^2\theta\, d\theta$$

which equals

$$(1/2)\int_0^{\pi/4} \sec^3\theta\, d\theta$$

The formula for this would be given to you:

$$\int \sec^3\theta\, d\theta = (1/2)\sec\theta\tan\theta + (1/2)\ln|\sec\theta + \tan\theta| + C$$

Use this formula to find the final answer.

(3) $f(x) = (2/3)x^{3/2}$ from $x = 0$ to $x = 1$.

Answer:

$$\sqrt{1 + (f'(x))^2} = \sqrt{1 + x}$$

Hence, the arc length is

$$\int_0^1 \sqrt{1+x}\,dx = (1+x)^{3/2}\Big|_0^1 = \sqrt[3]{4} - 1$$

Problem 4: Find the following volumes

(1) Very Loud

(2) Very soft

(3) The object obtained by placing the base of equilateral triangles on a circle of radius 1.

Answer: It is easier to consider half of this object (draw a picture!). If an equilateral triangle has base of length s it has area $(\sqrt{3}/4)s^2$. The upper half of the unit circle is given by $y = \sqrt{1 - x^2}$. So at point $x \in [-1, 1]$ the triangle at that point has area $A(x) = \frac{\sqrt{3}(1-x^2)}{4}$. Integrate this to find half the volume:

$$\int_0^1 \frac{\sqrt{3}}{4}(1 - x^2)\,dx = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{12}.$$

The total volume is twice this amount.

(4) The object obtained by rotating the graph of $f(x) = xe^x$ for $x \in [1, 2]$ around the x axis.

Answer: The volume is given by the integral:

$$\int_1^2 \pi x^2 e^{2x}\,dx$$

Use integration by parts twice.

(5) The object obtained by rotating the graph of $f(x) = xe^x$ for $x \in [1, 2]$ around the y axis.

Answer: Solve the integral:

$$\int_1^2 2\pi x^2 e^x \, dx$$

(6) The object obtained by rotating the region between $f(x) = -x^2 + 4$ and $g(x) = x^2 + 2$ around the x axis.

Answer: Solve the integral:

$$\int_{-1}^{1} (-2x^2 + 2)^2 \, dx$$

You will need to expand the integrand.

Problem 5: Solve the following work problems:

(1) Suppose that a bucket which weighs 3 kg is attached to a 30 meter long rope which weighs a total of 300 kg. If the rope is hanging off a 30 meter tall building, how much work is required to haul the rope (with bucket attached) to the top of the building?

Answer: The work required to move the bucket is $3 \cdot 9.8 \cdot 30$ J. We now consider just the rope.
A section of rope at height y_i^* of length Δy is weighs $(300kg)\Delta y/30m = 10\Delta y$ kg. The force required to move this is $98\Delta y$ N. It moves a distance of $30 - y_i^*$. The work to move this piece of rope is $98(30 - y_i^*)\Delta y$.
Adding up all the pieces of rope we find that the work is approximately:

$$\sum_{i=1}^{n} 98(30 - y_i^*)\Delta y$$

To find the actual work take the limit as $n \to \infty$ and change the result to an integral:

$$\int_0^{30} 98(30 - y) \, dy = -49(30 - y)^2 \Big|_0^{30} = 49(30)$$

So the total amount of work for rope and bucket is 2352 J.

(2) The lower half of a sphere of radius 4 m is buried in the ground so that its top is level with the ground. The tank has water in it, so that the water is 3 m deep at

its deepest point. How much work is required to pump the water out the top of the tank? (Water has a density of 1 g/cm^3).

Answer: Say that the ground is at height zero and that the center of the sphere is at the origin. A slab of water at height y_i^* of thickness Δy has volume $\pi(16 - y_i^{*2})\Delta y$. Its mass is $\pi(16 - y_i^{*2})\Delta y \cdot 1000$. It moves a distance of $0 - y_i^*$ and so the total work to move the slab is:

$$980\pi(16 - y_i^{*2})(-y_i^*)\Delta y$$

Add up, take the limit, and change to an integral as before:

$$\int_{-4}^{-1} 980\pi(16 - y^2)(4 - y)\, dy$$

Finally, solve the integral.

Problem 6: Find all solutions to the following differential equations:

(1) $y' = 3y$

Answer: $y = Ae^{3t}$

(2) $y' = yt$

Answer: $y = Ae^{t^2/2}$

(3) $y' = (y^2 + 1)(t^2 + 1)$

Answer: $y = \tan\left((1/3)t^3 + t + C\right)$

(4) $y' = (1/y)\ln t$

Answer: $(1/2)y^2 = t(\ln t - 1) + C$

Problem 7: Perform the following integrations:

(1) $\int_0^1 x\cos(x)\, dx$

(2) $\int_3^4 \sin^3(x)\, dx$

(3) $\int x^3 \ln(x)\, dx$

(4) $\int x \arctan(x)\, dx$

(5) $\int \sin(x) e^x\, dx$

Answer: For the second integral use a trig identity. For all the others use integration by parts.

Problem 8: Find the following antiderivatives.

(1) $\int \frac{-2x^2+4}{(2x+1)(x^2+x+1)}\, dx$

Answer: Use the method of partial fractions to find

$$\frac{-2x^2+4}{(2x+1)(x^2+x+1)} = \frac{14/3}{2x+1} + \frac{(-10/3)x-(2/3)}{x^2+x+1}$$

We will integrate each piece separately:

$$(14/3)\int \frac{1}{2x+1}\, dx = (14/3)(1/2)\ln\left|2x+1\right| + C$$

For the other fraction use completing the square to write

$$x^2+x+1 = (x+(1/2))^2 + (3/4)$$

Substitute $u = x + (1/2)$ to get the integral:

$$\int \frac{(-10/3)(u-(1/2))-(2/3)}{u^2+(3/4)}\, du$$

Rewrite as:

$$(-10/3)\int \frac{u}{u^2+(3/4)}\, du - \int \frac{1}{u^2+(3/4)}\, du$$

To solve the first integral, substitute $v = u^2 + (3/4)$ to get the integral

$$(-10/3)(1/2)\int \frac{1}{v}\, dv = (-5/3)\ln\left|x^2+x+1\right| + C$$

To solve the second integral, factor a $(3/4)$ out to get:

$$(4/3)\int \frac{1}{(\sqrt{(4/3)}u)^2+1}\, du$$

Substitute $v = (4/3)u$ to obtain:

$$\int \frac{1}{v^2+1}\, dv = \arctan(x+(1/2)) + C$$

Hence the answer to the problem is:

$$(7/3)\ln\left|2x+1\right| - (5/3)\ln\left|x^2+x+1\right| + \arctan(x+(1/2)) + C$$

(2) $\int \frac{8x^2+19x-26}{(x+3)^2(4x+1)}\, dx$

Answer: The method of partial fractions allows us to rewrite the integral as:

$$\int \frac{3}{x+3} + \frac{1}{(x+3)^2} - \frac{(4)}{4x+1}\, dx$$

This equals:

$$3\ln|x+3| - \frac{1}{(x+3)} - 4\ln|4x+1| + C$$

Problem 9: Find the following antiderivatives.

(1) $\int \frac{1}{x^2\sqrt{x^2-9}}\, dx$

Answer: Trig substitution gives us:

$$\int \frac{\cos^3\theta}{\sin\theta}\, d\theta = \int \frac{(1-\sin^2\theta)\cos\theta}{\sin\theta}\, d\theta$$

Substitute $u = \sin\theta$ to get:

$$\int \frac{(1-u^2)}{u}\, du = \ln|u| - \frac{1}{2}u^2 + C$$

Plugging back in gives:

$$\ln|\sin\theta| - \frac{1}{2}\sin^2\theta + C$$

We know $x = \sec\theta$ and so using a triangle and the Pythagorean theorem we get $\sin\theta = \frac{\sqrt{x^2-1}}{x}$ and so the answer is

$$\ln\left|\frac{\sqrt{x^2-1}}{x}\right| - \frac{1}{2}\left(\frac{x^2-1}{x^2}\right) + C$$

(2) $\int x\sqrt{x^2+4}\,dx$

Answer: Let $u = x^2 + 4$. The integral becomes

$$\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

Which is

$$\sqrt{u} + C = \sqrt{x^2+4} + C.$$

(3) $\int \frac{x^2}{\sqrt{4x-x^2}}\,dx$

Answer: Complete the square and rewrite the integral as:

$$\int \frac{x^2}{\sqrt{4-(x-2)^2}}\,dx$$

Let $u = x - 2$:

$$\int \frac{(u+2)^2}{\sqrt{4-u^2}}\,du$$

Let $u = 2\sin\theta$

$$\int \frac{(2+2\sin\theta)^2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,d\theta$$

which equals

$$\int \frac{(2+2\sin\theta)^2\cos\theta}{2\cos\theta}\,d\theta = 2\int 1+\sin^2\theta\,d\theta$$

which equals:

$$2\left[\theta + \int (1/2)(1-\cos(2\theta))\,d\theta\right] = 3\theta - \frac{1}{2}\sin(2\theta) + C$$

This equals:

$$3\arcsin\left(\frac{x-2}{2}\right) - 2\sin\theta\cos\theta + C$$

Which equals

$$3\arcsin\left(\frac{x-2}{2}\right) - (x-2)\frac{\sqrt{4x-x^2}}{2} + C$$

Problem 10: Find the following antiderivatives, without using an antiderivative table.

(1) $\int e^{\sqrt[3]{x}}\,dx$

Answer: Let $u = \sqrt[3]{x}$. Then $du = (1/3)x^{-2/3}\,dx = (1/3)u^{-2}\,dx$. The integral is then:

$$\int 3u^2 e^u\,du$$

and you can solve this using integration by parts twice.

(2) $\int x\arcsin(x)\,dx$

Answer: Use integration by parts with $u = \arcsin(x)$ and $dv = x\,dx$.

(3) $\int \sin(4x)\cos(3x)\,dx$

Answer: Use the trig identities: $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ and $\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$ to find that:

$$\sin(a)\cos(b) = (1/2)[\sin(a+b) + \sin(a-b)]$$

which means:

$$\sin(4x)\cos(3x) = (1/2)[\sin(7x) + \sin(x)]$$

which you know how to integrate.

(4) $\int \ln(x^2-1)\,dx$

Answer: Write $x^2 - 1 = (x+1)(x-1)$ so that the integral becomes:

$$\int \ln(x+1) + \ln(x-1)\,dx$$

Then use the formula for the antiderivative of ln or solve it using integration by parts.

Problem 11: Write down the formulae for the Taylor polynomials $p_n(x)$ of e^x, $\sin x$, and $\cos x$ centered at $x_0 = 0$.

Problem 12:

(1) Find the Taylor polynomial $p_5(x)$ for $f(x) = \sqrt[3]{x}$ centered at $x_0 = 1$.

Answer:

$$1+\frac{1}{3}(x-1)+\frac{-2}{9\cdot 2}(x-1)^2+\frac{10}{27\cdot 3!}(x-1)^3+\frac{-80}{81\cdot 4!}(x-1)^4+\frac{880}{243\cdot 5!}(x-1)^5$$

(2) Find a bound on the error $|f(3/2) - p_5(3/2)|$ using Taylor's theorem.

Answer: We have

$$|f^{(6)}(x)| = \frac{14\cdot 880}{3^6}|x|^{-16/3} \leq 16$$

for $x \geq 1$. Choose, therefore, $K_6 = 16$. We then have:

$$|f(3/2) - p_5(3/2)| \leq \frac{16(1/2)^6}{6!}$$

Problem 13: Find the Taylor polynomial $p_3(x)$ for $f(x) = \arcsin(x)$ centered at $x_0 = 0$.

Answer:

$$p_3(x) = x + x^3/6$$

Problem 14: Determine whether or not the following improper integrals converge. If so, find what they converge to.

(1) $\int_0^\infty xe^{-x}\,dx$

Answer: Converges to 1.

(2) $\int_{-\infty}^\infty x\,dx$

Answer: Diverges

(3) $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Answer: Converges to π.

(4) $\int_0^1 x^{-1/2}\,dx$

Answer: Converges to 2.

(5) $\int_0^\infty \frac{1}{x(x+1)}\,dx$

Answer: Use the method of partial fractions to find:

$$\int_0^\infty \frac{1}{x(x+1)}\,dx = \Big[\ln|x| - \ln|x+1|\Big]_0^\infty$$

Use properties of natural log to rewrite it as:

$$\left[\ln\left|\frac{x}{x+1}\right|\right]_0^\infty = \ln\left|\frac{x}{x+1}\right|\Bigg|_0^1 + \ln\left|\frac{x}{x+1}\right|\Bigg|_1^\infty$$

We then have:

$$\lim_{s\to 0^+} \ln\left|\frac{s}{s+1}\right| = \ln\left|\lim_{s\to 0^+}\frac{s}{s+1}\right| = -\infty$$

Hence, the integral does not converge. Notice however that $\int_1^\infty \frac{1}{x(x+1)}\,dx$ does converge.

Problem 15: Determine if the following integrals converge or diverge.

(1) $\int_0^\infty \frac{1}{x^3+1}\,dx$

Answer: Write the integral as $\int_0^1 \frac{1}{x^3+1}\,dx + \int_1^\infty \frac{1}{x^3+1}\,dx$. The first integral is just a regular definite integral. The second integral converges by comparing the integrand to $\frac{1}{x^3}$. Hence, the integral converges.

(2) $\int_1^\infty \frac{\ln(x)}{x^2+1}\,dx$

Answer: Compare the integrand to $\frac{\ln(x)}{x^2}$ which has an antiderivative that you can find.

(3) $\int_1^\infty \sin\Big(1/x\Big)\,dx$

Answer: Perform the substitution $u = 1/x$ to rewrite the integral as:

$$\int_0^1 u^2 \sin(u)\,du$$

This is a definite integral of a continuous function and so the original integral converges.

(4) $\int_1^\infty \frac{(\cos(x)+5)^{15})}{x^3}\, dx$

Answer: Notice that $(\cos(x)+5)^{15} \leq 6^{15}$ and so:

$$\int_1^\infty \frac{(\cos(x)+5)^{15}}{x^3}\, dx \leq \int_1^\infty \frac{6^{15}}{x^3}\, dx$$

You can find an antiderivative for the second integrand and discover that by the comparison test the original integral converges.

(5) $\int_1^\infty \frac{(\cos(x)+5)^{15})}{x}\, dx$

Answer: Notice that $(\cos(x)+5)^{15} \geq 4^{15}$. By the comparison test

$$\int_1^\infty \frac{(\cos(x)+5)^{15}}{x}\, dx \geq \int_1^\infty \frac{4^{15}}{x}\, dx$$

and the last integral diverges, so the first one does as well.

(6) $\int_{-1}^1 \frac{\sqrt{1-x^2}}{x^2}\, dx$

Answer: This integral is improper because the integrand is discontinuous at $x = 0$. Consider therefore the integral:

$$\int_0^1 \frac{\sqrt{1-x^2}}{x^2}\, dx$$

When $0 < x \leq 1$, we have that $\frac{\sqrt{1-x^2}}{x^2} \geq \frac{1}{x}$. (Check!) Hence:

$$\int_0^1 \frac{\sqrt{1-x^2}}{x^2}\, dx \geq \int_0^1 \frac{1}{x}\, dx.$$

The last integral diverges and so our integral does as well.

Problem 16: Determine if the volume of the object obtained by rotating the graph of $y = x^{-1/2}\cos(x)$ for $x \in [1, \infty)$ around the x axis is finite.

Answer: Use the disc method. Consider the integral:

$$\int_1^\infty \pi \frac{\cos^2(x)}{x}\, dx$$

This integral can be rewritten using the identity:

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x))\, dx$$

Hence:

$$\int_1^\infty \frac{\cos^2(x)}{x}\,dx = \int_1^\infty \frac{1}{2x}\,dx + \int_1^\infty \frac{\sin(x)}{2x}\,dx$$

the first integral on the right hand side diverges. The second converges (but that's harder to see). So, the volume is not finite.

Problem 17: Determine the limits of the following sequences. You may assume that the limit exists.

(1) $\{\frac{f_{k+1}}{f_k}\}$ where f_k is the kth Fibonacci number.

Answer: Let L be the limit. Then:

$$L = \lim_{k\to\infty} \frac{f_k + f_{k-1}}{f_k} = 1 + \lim_{k\to\infty} \frac{f_{k-1}}{f_k} = 1 + \frac{1}{L}$$

Hence, $L^2 - L - 1 = 0$ and so $L = \frac{1+\sqrt{5}}{2}$ (using the fact that $L \geq 0$.

(2) $\{\frac{2^k}{k!}\}$ **Answer:** One method is to notice that by the ratio test

$$\sum_{k=1}^{\infty} \frac{2^k}{k!}$$

converges. However an infinite series converges only if the sequence of its terms goes to zero. Hence $\lim_{k\to\infty} \frac{2^k}{k!} = 0$.

(3) $\{a_k\}$ where $a_1 = \sqrt{3}$ and $a_k = \sqrt{3 + a_{k-1}}$.

Answer: Let L be the limit. Then $L = \sqrt{3+L}$ and so $L^2 - L - 3 = 0$. Hence, $L = \frac{1+\sqrt{13}}{2}$.

(4) $\{a_k\}$ where $a_1 = 2$ and $a_k = 2 + \frac{1}{a_{k-1}}$

Answer: Let L be the limit. Notice that $L = 2 + \frac{1}{L}$. Hence $L^2 - 2L - 1 = 0$ and so $L = \frac{2+\sqrt{8}}{2} = 1 + \sqrt{2}$.

Problem 18: What is $\sum_{k=0}^{1}(2/3)^k$?

Answer: The answer is 3 by the formula for geometric series.

Problem 19: Determine whether the following series converge or diverge. Be sure to explain your reasoning.

(1) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

Answer: Diverges by the integral test

(2) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^2}}$

Answer: Diverges by the integral test.

(3) $\sum_{k=1}^{\infty} k/2^k$

Answer: Converges by the ratio test.

(4) $\sum_{k=1}^{\infty} \frac{k}{k^3+1}$

Answer: Converges by comparison test with $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Problem 20: Determine whether the following series converge or diverge.

(1) $\sum_{k=1}^{\infty} \frac{1}{f_k}$ where f_k is the kth Fibonacci number.

Answer: Converges by the ratio test (see class notes).

(2) $\sum_{k=1}^{\infty} \frac{3^k}{k!}$

Answer: Converges by the ratio test.

(3) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k!}}$

Answer: Apply the ratio test and consider:

$$\lim_{k\to\infty} \frac{\sqrt{(k)!}}{\sqrt{(k+1)!}} = \lim_{k\to\infty} \sqrt{\frac{k!}{(k+1)!}} = \lim_{k\to\infty} \sqrt{\frac{1}{k+1}} = 0$$

(4) $\sum_{k=1}^{\infty} k^{-k}$

Answer: Apply the ratio test and consider:

$$\begin{aligned}\lim_{k\to\infty}\frac{k^k}{(k+1)^{k+1}} &= \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k\frac{1}{k+1}\\ &= \left(\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k\right)\left(\lim_{k\to\infty}\frac{1}{k+1}\right)\\ &= e^{-1}\cdot 0\\ &= 0\end{aligned}$$

Problem 21: Explain why the following series converge.

(1) $\sum_{k=1}^{\infty}(-1)^k\frac{1}{k}$

Answer: Use the alternating series test.

(2) $1+1/4-1/9-1/16+1/25+1/36-1/49-1/81+\ldots$

Answer: Write the series as $\sum_{k=1}^{\infty}a_k$ and notice that $a_k=\pm\frac{1}{k^2}$. Since $\sum_{k=1}^{\infty}|a_k|=\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by the integral test, the absolute convergence theorem tells us that our series also converges.

(3) $\sum_{k=1}^{\infty}(-3/4)^k$

Answer: This converges by either the alternating series test or by the fact that it is a geometric series with $r<1$.

Problem 22: Determine the radius of convergence for the following power series. Except for (4), also find the interval of convergence.

(1) $\sum_{k=1}^{\infty}2^kx^k$

Answer: Write as a geometric series: $\sum_{k=1}^{\infty}(2x)^k$ and recall that it will converge when $2|x|<1$. I.e. $|x|<(1/2)$. The radius of convergence is, therefore, (1/2).

The interval of convergence is $(-\frac{1}{2}$ to $\frac{1}{2})$. Be sure you understand why the endpoints are not included.

(2) $\sum_{k=1}^{\infty} kx^k$

Answer: By the ratio test, this converges when $|x| < 1$. The radius of convergence is, therefore, 1. The interval of convergence is $(-1, 1)$ since the series $\sum_{k=1}^{\infty}(-1)^k k$ and $\sum_{k=1}^{\infty} k$ don't converge.

(3) $\sum_{k=1}^{\infty} k!x^k$

Answer: By the ratio test this converges when

$$\lim_{k\to\infty} \frac{(k+1)!}{k!}|x| = \lim_{k\to\infty} (k+1)|x| < 1$$

However, when $x \neq 0$ the above limit is ∞ and so the radius of convergence is 0 and the interval of convergence is $\{0\}$.

(4) $\sum_{k=1}^{\infty} f_k x^k$ where f_k is the kth Fibonacci number

Answer: Use the ratio test:

$$\lim_{k\to\infty} \frac{f_{k+1}}{f_k}|x|$$

Now, as shown in class $\lim_{k\to\infty} \frac{f_{k+1}}{f_k} = \phi$, the golden ratio. Hence $(1/\phi)$ is the radius of convergence.

(5) $\sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}$

Answer: By the ratio test (and some work) this has radius of convergence equal to 1. When $x = 1$ we have the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$. By the comparison test we have:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

So the series converges when $x = 1$. The absolute convergence test shows that it also converges when $x = -1$. The interval of convergence is, therefore $[-1, 1]$.

Problem 23: Find power series representations for the following functions at the point indicated. State the radius of convergence of the power series.

(1) $f(x) = e^x \qquad x_0 = 0$

Answer: $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ The radius of convergence is ∞.

(2) $f(x) = \frac{1}{2x+1} \qquad x_0 = 1$

Answer: It's probably easiest to use Taylor polynomials. Some work will show that the nth Taylor polynomial has the form:

$$p_n(x) = \sum_{k=0}^{n} \frac{(-1)^k \cdot 2^k}{3^{k+1}}(x-1)^k$$

The power series is thus:

$$\frac{1}{2x+1} = \sum_{k=1}^{\infty} \frac{(-1)^k \cdot 2^k}{3^{k+1}}(x-1)^k$$

Use the ratio test to find that the series converges when:

$$\lim_{k\to\infty} \frac{2}{3}|x-1| < 1$$

That is, when $|x| < (3/2)$. So the radius of convergence is (3/2).

(3) $f(x) = \ln x \qquad x_0 = 1$

Answer: Consider the power series for $\frac{1}{1-x}$:

$$\sum_{k=0}^{\infty} x^k$$

Replace x with $1-x$ to obtain the power series

$$\sum_{k=0}^{\infty} (-1)^k (x-1)^k$$

for $(1/x)$. Integrate to find the power series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}(x-1)^{k+1}$$

for $f(x) = \ln x$. The radius of convergence is 1.

(4) $f(x) = \arctan(x) \qquad x_0 = 0$

Answer: Use the power series $\frac{1}{1-x} = \sum x^k$ and substitute in $x \to -x^2$ to get the power series:

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

Now integrate to get:

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} x^{2k+1}$$

The radius of convergence is 1.

(5) $f(x) = e^{x^2} \qquad x_0 = 0$

Answer: Make the substitution $x \to x^2$ into the power series for e^x. The radius of convergence is ∞.

(6) $f(x) = \sin(x^3) \qquad x_0 = 0$

Answer: Make the substitution $x \to x^3$ into the power series for $\sin(x)$. The radius of convergence is ∞.

(7) $f(x) = \frac{1}{1+x^3} \qquad x_0 = 0$

Answer: Make the substitution $x \to -x^3$ into the power series for $\frac{1}{1-x}$. The radius of convergence is 1.

Problem 24: Find power series representatives based at the point indicated for an **antiderivative** of the function $f(x)$.

(1) $f(x) = x^2 e^{x^2}$ at $x_0 = 0$

Answer: In the previous problem you should have found that:

$$e^{x^2} = \sum_{k=0}^{\infty} x^{2k}/k!$$

Multiply by x^2 to find:

$$x^2e^{x^2} = \sum_{k=0}^{\infty} x^{2k+2}/k!$$

Now integrate:

$$\int x^2e^{x^2} = \sum_{k=0}^{\infty} \frac{x^{2k+3}}{k!(2k+3)} + C$$

(2) $f(x) = \sin(x^3)$ at $x_0 = 0$

Answer:

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{6k+4}}{(2k+1)!(6k+4)}$$

(3) $f(x) = \frac{1}{1+x^3}$ at $x_0 = 0$

Answer:

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{3k+1}$$

Problem 25: The following are a list of statements that you may be asked to prove. The numbers involved may change.

(1) If $f'(x) \leq 10$ for $x \in [a,b]$ then the nth left hand approximation L_n to $I = \int_a^b f(x)\,dx$ satisfies

$$I - L_n \leq \frac{10(b-a)^2}{2n}$$

(2) If $f^{(3)}(x) \leq 18$ then the 2nd MacLaurin approximation $p_2(x)$ satisifies

$$f(5) - p(5) \leq \frac{18 \cdot 5^3}{3!}$$

(3) Suppose that $r > 0$, prove that $\sum_{k=0}^{\infty} r^k$ converges if and only if $r < 1$ and that when it converges it equals $1/(1-r)$.

Answer: It is a fact that

$$1 + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Hence the nth partial sum of the series is:

$$S_n = \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

If $r > 1$ we have $\lim_{n\to\infty} r^{n+1} = \infty$. If $r < 1$ we have $\lim_{n\to\infty} r^{n+1} = 0$. Hence:

$$\lim_{n\to\infty} S_n = \begin{cases} -\infty & r > 1 \\ \frac{1}{1-r} & r < 1 \end{cases}$$

If $r = 1$, the series is $\sum_{k=0}^{\infty} 1 = \infty$. Hence, the geometric series converges if and only if $r < 1$.

(4) Prove that the sequence $\{a_k\}$ where $a_1 = 2$, $a_k = \sqrt{a_{k-1}}$ is a bounded, decreasing sequence.

(5) Prove that the sequence $\{a_k\}$ where $a_1 = \sqrt{2}$ and $a_k = \sqrt{2 + a_{k-1}}$ is a bounded, increasing sequence.

(6) Prove that if f is continuous on $[0, \infty)$ then if $\int_0^\infty |f(x)|\, dx$ converges so does $\int_0^\infty f(x)\, dx$

(7) Use a geometric series to explain why $.9999\ldots = 1$

(8) Let $\{a_k\}$ be a sequence of numbers all of which are whole numbers between 0 and 9 (including 0 and 9). Use the comparison test and a geometric series to explain why $\sum_{k=1}^{\infty} a_k/10^k$ converges. This partially explains why there are numbers with infinite, non-repeating decimal expansions.

Answer: Notice that $a_k \leq 9$ for all k. Hence, by the comparison test:

$$\sum_{k=1}^{\infty} a_k/10^k \leq \sum_{k=1}^{\infty} 9/10^k = 9\sum_{k=1}^{\infty}(1/10)^k = 10.$$