Math 10: Practice Exam 1 Answers and Solutions

Name:

You may use a calculator on the exam and a 3" ×5" notecard with notes on it. You have 65 minutes for the exam. Show all of your work. Your work **is** your answer.

Problem 1: Consider the definite integral $I = \int_e^{4e} \ln(x^2 + x)\ dx$.

(a) What is the Left Hand approximation to I with $n = 3$?

Answer:

$$e\ln(e^2 + e) + e\ln(4e^2 + 2e) + e\ln(9e^2 + e)$$

(b) Calculate the first derivative of $f(x) = \ln(x^2 + x)$ and use this to show that L_3 is an underestimate of I.

Answer: $f'(x) = \frac{2x+1}{x^2+x}$. On the interval $[e, 4e]$ all of these numbers are positive. Since $f'(x)$ is positive, $f(x)$ is increasing. Since $f(x)$ is increasing, the left hand approximiation underestimates the integral.

(c) What is an upperbound on $|I - L_3|$?

Answer: Notice that:

$$|f'(x)| \le 2x + 1 \le 8e + 1$$

since on $[e, 4e]$ $x^2 + x \ge 1$ and since $2x + 1$ is increasing.

Thus, from the formula for an error bound:

$$|I - L_3| \le \frac{(8e + 1)(3e)^2}{6}$$

(d) What value of n would be needed in order to assure that

$$|I - L_n| \leq .001?$$

Answer:

$$|I - L_n| \leq \frac{(8e+1)(3e)^2}{2n} \leq .001$$

Thus,

$$n \geq \frac{(8e+1)(3e)^2}{.002}$$

(e) Calculate the midpoint approximation to I with $n = 3$.

Answer:

$$e[\ln((1.5e)^2 + 1.5e) + \ln((2.5e)^2 + 2.5e) + \ln((3.5e)^2 + 3.5e)]$$

(f) What is an upperbound on $|I - M_3|$?

$$|f''(x)| = \frac{|2(x^2+x) - (2x+1)^2|}{(x^2+x)^2} \leq 2x^2 + 2x \leq (4e)^2 + 8e$$

Thus,

$$|I - M_3| \leq \frac{(16e^2 + 8e)(3e)^3}{216}$$

Problem 2: Let $f(x)$ be any continuous function on the interval $[2, 3]$ such that $f'(x) \leq 18$. Show that $f(x) \leq 18(x-2) + f(2)$.

Answer:

$$\begin{aligned}
f'(x) &\leq 18 \\
\int_2^x f'(t)\,dt &\leq \int_2^x 18\,dt \\
f(x) - f(2) &\leq 18(x-2) \\
f(x) &\leq 18(x-2) + f(2)
\end{aligned}$$

Problem 3: Consider the initial value problem:

$$y' = y^2 \qquad y(0) = 1$$

(a) Use Euler's method with three steps of size 0.2 to estimate $y(0.6)$.

Answer:

$$\begin{aligned} y_0 &= 1 \\ y_1 &= 1(.2) + 1 &= 1.2 \\ y_2 &= (1.2)^2(.2) + 1.2 &= 1.488 \\ y_3 &= (1.488)^2(.2) + 1.488 &= 1.9308288 \end{aligned}$$

$$y(.6) \approx y_3 = 1.9308288$$

(b) Show that the exact solution of the IVP is $y(t) = \frac{1}{1-t}$.

Answer:

$$y'(t) = (-1)\frac{-1}{(1-t)^2} = (y(t))^2$$

And, $y(0) = 1$.

Problem 4: Compute the area contained in the region bounded by:

$$\begin{aligned} x &= 1 \\ x &= 4 \\ y &= -(x-3)^2 + 4 \\ y &= \ln(x) \end{aligned}$$

Answer:

$$-\frac{1}{3} + 4 - \frac{8}{3} + 8 - 4\ln 4 + 3 = 12 - 4\ln 4$$

Problem 5: Write down an integral representing the arclength of the curve

$$y = \arcsin(x)$$

from the point $(0, 0)$ to the point $(1, \pi/2)$. You should actually work out the derivative. You do not need to solve the integral.

Answer:

$$\int_0^1 \sqrt{1 + \frac{1}{1 - x^2}}\, dx$$

Problem 6: Compute the volume of an object which has a base that is a right triangle with sides of length 3, 4, and 5 as indicated in the figure. The cross-sections are squares. See the figure.

Answer: 16

Problem 7: Compute the volume of the object obtained by rotating the region bounded by

$$\begin{aligned} y &= \sin(x) \\ y &= 0 \\ x &= 0 \\ x &= \pi \end{aligned}$$

around the x-axis. (Hint: Use the trig identity: $\cos(2x) = 1 - 2\sin^2(x)$.)

Answer: $\pi^2/2$

Problem 8: Compute the volume of the object obtained by rotating the region bounded by:

$$\begin{aligned} y &= \sin(x^2) \\ y &= 0 \\ x &= 0 \\ x &= 1 \end{aligned}$$

around the y axis.

Answer: $\pi - \pi \cos(1)$

Problem 9: A 10 meter rope which weighs 2 kg/m is hanging off the top of a 10 meter building. How much work is done in hauling the rope to the top of the building? (The acceleration due to gravity is $g = -9.8$ m/s^2.)

Answer: $100(9.8) = 980$ J

Problem 10: Find all solutions to the DE: $y' = t(y-1)$.

Answer: $y = Ae^{t^2/2} + 1$

Problem 11: Integrate the following:

(a)

$$\int x^2 e^x \, dx$$

Answer: $x^2 - 2xe^x + 2e^2 + C$

(b)

$$\int xe^{x^2} \, dx$$

Answer: $\frac{1}{2}e^{x^2} + C$

(c)

$$\int_0^1 x \sin x \; dx$$

Answer: $\sin(1) - \cos(1)$

(d)

$$\int (\sin x)e^x \; dx$$

Answer: $\frac{1}{2}e^x(\sin x - \cos x)$

(e)

$$\int_1^2 \frac{5x+2}{(x+4)(2x-1)} \; dx$$

Answer:

(f)

$$\int \frac{2x^3 + 7x^2 + 44x + 58}{(x+3)^2(x^2+4)}$$

Answer: