Math 10: Practice Exam 2

Name:

You may use a calculator on the exam and a 3" × 5" notecard with notes on it. You have 65 minutes for the exam. Show all of your work. Your work **is** your answer. The practice exam is longer and more difficult than the actual exam.

Problem 1: Solve the following integrals:

(1)

$$\int_{1}^{2\sqrt{2}-1} \sqrt{x^2 + 2x - 3}\, dx$$

You may need the formulae:

$$\begin{aligned} \int \sec\theta\, d\theta &= \ln|\sec\theta + \tan\theta| + C \\ \int \sec^3\theta\, d\theta &= \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C \end{aligned}$$

(2)

$$\int_{1}^{2} \frac{e^x}{1 - e^x}\, dx$$

(3)

$$\int \frac{\ln(x+1)}{\sqrt{x}}\, dx$$

Problem 2: Find the formula for the Taylor polynomial $p_n(x)$ of $f(x) = e^x$ at $x_0 = 0$. Show all of your work.

Problem 3: Find the formula for the Taylor polynomial $p_n(x)$ of $f(x) = \sin(x)$ at $x_0 = 0$. Show all of your work.

Problem 4: Find the formula for the Taylor polynomial $p_n(x)$ of $f(x) = \cos(x)$ at $x_0 = 0$. Show all of your work.

Problem 5:

(1) Find the Taylor polynomial $p_3(x)$ of $f(x) = \arctan(x)$ at $x_0 = 0$.

(2) Use Taylor's theorem to find a bound on the error $|f(1) - p_3(1)|$. (Hint: For $x \geq 0$, $|f^{(4)}(x)| \leq 33x$.)

Problem 6:

(1) Find the Taylor polynomial $p_3(x)$ of $f(x) = \ln(x)$ at $x_0 = 1$.

(2) Use Taylor's theorem to find a bound on the error $|f(2) - p_3(2)|$.

Problem 7:

(1) Find the Taylor polynomial $p_3(x)$ of $f(x) = \ln^2(x)$ at $x_0 = 1$.

(2) Use Taylor's theorem to find a bound on the error $|f(3) - p_3(3)|$. Hint: $|f^{(4)}(x)| \leq 46$ for $x \in [1,3]$.

Problem 8: Suppose that $f(x)$ is some C^∞ function and that $p_n(x)$ is its nth Taylor polynomial at $x_0 = 0$. Suppose that b is a real number. Prove that the Taylor polynomial for the function $h(x) = f(x-b)$ at $x_0 = b$ is equal to $p_n(x-b)$.

Problem 9: Suppose that $f(x)$ is an odd C^∞ function. Prove that the nth Taylor polynomial $p_n(x)$ at $x_0 = 0$ for $f(x)$ has only terms with odd powers of x.

Problem 10: Find the 3rd Fourier polynomial $q_3(x)$ for the function $f(x) = x\sin(x)$ for $x \in [-\pi, \pi]$. You may wish to use the formulae:

$$\int \sin(ax)\sin(bx)\,dx = \frac{\sin\big((a-b)x\big)}{2(a-b)} - \frac{\sin\big((a+b)x\big)}{2(a+b)} + C \qquad \text{when } a \neq b$$

$$\int \sin(ax)\cos(bx)\,dx = -\frac{\cos\big((a-b)x\big)}{2(a-b)} - \frac{\cos\big((a+b)x\big)}{2(a+b)} + C \qquad \text{when } a \neq b$$

Problem 11: Calculate the following improper integrals. If they diverge say so.

(1)

$$\int_{-2}^{2} \frac{3x^2 + 4x - 3}{\sqrt{|x(x-1)(x+3)|}}\, dx$$

(2)

$$\int_{-\infty}^{0} \frac{1}{2x - 5}\, dx$$

(3)

$$\int_{-1}^{1} \frac{e^x}{e^x - 1}\, dx$$

(4)

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx$$

Problem 12: Determine whether the following integrals converge or diverge. Be sure to give a complete explanation.

(1)

$$\int_{0}^{\pi} \sin\left(\frac{1}{x}\right) dx$$

(2)

$$\int_{0}^{1} \frac{\sin(x)}{x}\, dx$$

(3)

$$\int_{0}^{\infty} \frac{\cos^2(x)}{1 + x^2}\, dx$$

(4)

$$\int_3^\infty \frac{\ln(x^2+2)}{x}\,dx$$

Problem 13: Prove that if $f(x)$ is a continuous function such that $\int_0^\infty |f(x)|\,dx$ converges then $\int_0^\infty f(x)\,dx$ also converges.

Problem 14: Consider the graph of $f(x) = x^{-3/2}$ for $x \in [1, \infty)$.

(1) Determine whether or not the solid obtained by rotating the region between the graph and the x axis around the x axis has finite volume.

(2) Determine whether or not the solid obtained by rotating the region between the graph and the x axis around the y axis has finite volume.

Problem 15: The normal curve is the function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ for $x \in (-\infty, \infty)$. Show that the area between the graph of $f(x)$ and the x-axis is finite.