Math 10: Practice Exam 2

Name:

You may use a calculator on the exam and a 3” × 5” notecard with notes on it. You have 65 minutes for the exam. Show all of your work. Your work **is** your answer. The practice exam is longer and more difficult than the actual exam.

Problem 1: Solve the following integrals:

(1)

$$\int_{2}^{2\sqrt{2}} \sqrt{x^2+2x-3}\,dx$$

You may need the formulae:

$$\begin{aligned} \int \sec\theta\,d\theta &= \ln|\sec\theta+\tan\theta|+C \\ \int \sec^3\theta\,d\theta &= \tfrac{1}{2}\sec\theta\tan\theta+\tfrac{1}{2}\ln|\sec\theta+\tan\theta|+C \end{aligned}$$

Answer:

$$\begin{aligned} \int_1^{2\sqrt{2}-1} \sqrt{x^2+2x-3}\,dx &= \\ \int_1^{2\sqrt{2}-1} \sqrt{(x+1)^2-4}\,dx &= \\ \int_2^{2\sqrt{2}} \sqrt{t^2-4}\,dt &= \\ \int_0^{\pi/4} \sqrt{4\tan^2\theta}(2\sec\theta\tan\theta)\,d\theta &= \\ \int_0^{\pi/4} 4\sec\theta\tan^2\theta\,d\theta &= \\ 4\int_0^{\pi/4} \sec^3\theta-\sec\theta\,d\theta &= \\ 2\sqrt{2}-2\ln\left|\sqrt{2}+1\right| & \end{aligned}$$

(2)

$$\int_1^2 \frac{e^x}{1-e^x}\,dx$$

Answer: After a substitution the integral is:

$$\int_{1-e}^{1-e^2} \frac{-1}{u}\, du$$

which equals

$$-\ln|1-e^2| + \ln|1-e|.$$

A small simplification gives

$$\ln\left(\frac{1}{1+e}\right).$$

(3)

$$\int \frac{\ln(x+1)}{\sqrt{x}}\, dx$$

Answer: Integrating by parts gives:

$$2\sqrt{x}\ln(x+1) - \frac{1}{2}\int \frac{\sqrt{x}}{x+1}\, dx$$

In the integral do a substitution to find:

$$\int \frac{\sqrt{x}}{x+1}\, dx = \int \frac{2u^2}{u^2+1}\, du$$

Now long division shows that:

$$\int \frac{2u^2}{u^2+1}\, du = 2\Big(u - \arctan(u)\Big) + C$$

That is:

$$\int \frac{\ln(x+1)}{\sqrt{x}}\, dx = 2\sqrt{x}\ln(x+1) - \sqrt{x} + \arctan(\sqrt{x}) + C$$

Problem 2: Find the formula for the Taylor polynomial $p_n(x)$ of $f(x) = e^x$ at $x_0 = 0$. Show all of your work.

Answer:

$$p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Problem 3: Find the formula for the Taylor polynomial $p_n(x)$ of $f(x) = \sin(x)$ at $x_0 = 0$. Show all of your work.

Answer:

$$p_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Problem 4: Find the formula for the Taylor polynomial $p_n(x)$ of $f(x) = \cos(x)$ at $x_0 = 0$. Show all of your work.

Answer:

$$p_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$$

Problem 5:

(1) Find the Taylor polynomial $p_3(x)$ of $f(x) = \arctan(x)$ at $x_0 = 0$.

Answer:

$$p_3(x) = x - \frac{x^3}{3}$$

(2) Use Taylor's theorem to find a bound on the error $|f(1) - p_3(1)|$. (Hint: For $x \geq 0$, $|f^{(4)}(x)| \leq 33x$.)

Answer: Choose $K_4 = 33$. Taylor's theorem then guarantees:

$$|f(1) - p_3(1)| \leq \frac{33(1-0)^4}{4!} = \frac{33}{24}$$

Problem 6:

(1) Find the Taylor polynomial $p_3(x)$ of $f(x) = \ln(x)$ at $x_0 = 1$.

Answer:

$$p_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

(2) Use Taylor's theorem to find a bound on the error $|f(2) - p_3(2)|$.

Answer: Calculate $|f^{(4)}(x)| = \frac{6}{x^4} \leq 6$ when $x \geq 1$. Therefore, choose $K_4 = 6$. Taylor's theorem then guarantees that:

$$|f(2) - p_3(2)| \leq \frac{6(2-1)^4}{4!} = \frac{1}{4}$$

Problem 7:

(1) Find the Taylor polynomial $p_3(x)$ of $f(x) = \ln^2(x)$ at $x_0 = 1$.

Answer:

$$p_3(x) = (x-1)^2 - (x-1)^3$$

(2) Use Taylor's theorem to find a bound on the error $|f(3) - p_3(3)|$. Hint: $|f^{(4)}(x)| \leq 46$ for $x \in [1,3]$.

Answer: Taylor's theorem guarantees that $|f(3) - p_3(3)| \leq \frac{46(2^4)}{4!}$.

Problem 8: Suppose that $f(x)$ is some C^∞ function and that $p_n(x)$ is its nth Taylor polynomial at $x_0 = 0$. Suppose that b is a real number. Prove that the Taylor polynomial for the function $h(x) = f(x-b)$ at $x_0 = b$ is equal to $p_n(x-b)$.

Answer: The nth Taylor polynomial $r_n(x)$ for $h(x)$ is of the form:

$$r_n(x) = \sum_{k=0}^{n} \frac{h^{(k)}(b)}{k!}(x-b)^k.$$

The kth derivative of h is equal to $f^{(k)}(x-b)$ since the derivative of $x-b$ is just one. Thus $h^{(k)}(b) = f^{(k)}(b-b) = f^{(k)}(0)$. The coefficients of $r_n(x)$ are then exactly the same as the coefficients for $p_n(x)$ and so $r_n(x) = p_n(x-b)$.

Problem 9: Suppose that $f(x)$ is an odd C^∞ function. Prove that the nth Taylor polynomial $p_n(x)$ at $x_0 = 0$ for $f(x)$ has only terms with odd powers of x.

Answer: Since $f(x)$ is an odd function $f(-x) = -f(x)$ for all values of x. Define $g(x) = f(-x)$. The Taylor polynomial $r_n(x)$ for $g(x)$ is then exactly equal to $-p_n(x)$.

However, we can also obtain it by plugging $-x$ into $p_n(x)$. Hence: $-p_n(x) = p_n(-x)$. Suppose that $p_n(x) = \sum_{k=0}^{n} a_k x^n$. Then:

$$-\sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} a_k (-x)^k$$

Rewrite this as:

$$\sum_{k=0}^{n} (-a_k) x^k = \sum_{k=0}^{n} (-1)^k a_k x^k$$

Two polynomials are equal only if the coefficients are like terms are equal. In other words, for each k:

$$-a_k = (-1)^k a_k$$

But for even values of k, $(-1)^k a_k = a_k$. And so when k is even:

$$-a_k = a_k$$

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This only happens when $a_k = 0$. Hence, the coefficients of all the even terms are zero and so $p_n(x)$ has only odd powers of x. □

Problem 10: Find the 3rd Fourier polynomial $q_3(x)$ for the function $f(x) = x\sin(x)$ for $x \in [-\pi, \pi]$. You may wish to use the formulae:

$$\int \sin(ax)\sin(bx)\,dx = \frac{\sin\left((a-b)x\right)}{2(a-b)} - \frac{\sin\left((a+b)x\right)}{2(a+b)} + C \qquad \text{when } a \neq b$$

$$\int \sin(ax)\cos(bx)\,dx = -\frac{\cos\left((a-b)x\right)}{2(a-b)} - \frac{\cos\left((a+b)x\right)}{2(a+b)} + C \qquad \text{when } a \neq b$$

Answer: Begin by calculating the coefficients:

$$\begin{aligned}
a_0 &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} x\sin x\,dx = 1 \\
a_1 &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\,dx \\
&= \tfrac{1}{\pi}\int_{-\pi}^{\pi} \tfrac{1}{2}x\sin 2x\,dx = -\tfrac{1}{2} \\
a_k &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos(kx)\,dx \\
&= \tfrac{1}{\pi}\left[\tfrac{-x\cos((k-1)x)}{2(k-1)} - \tfrac{x\cos((k+1)x)}{2(k+1)}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \tfrac{\cos((k-1)x)}{2(k-1)} + \tfrac{\cos((k+1)x)}{2(k+1)}\,dx\right] \\
&= (-1)^{k+1}\left(\tfrac{1}{k-1} + \tfrac{1}{k+1}\right)
\end{aligned}$$

The last computation is valid when $k \geq 2$ and uses the fact that $\cos(n\pi) = 1$ when n is even and equals -1 when n is odd.

To compute the values for b_k notice that

$$y = x\sin x\sin(kx)$$

is an odd function since it is the product of three odd functions. This means that

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\sin(kx)\,dx = 0$$

for all values of k.

Thus, we have:

$$q_3(x) = 1 - \left(\frac{1}{2}\right)\cos x - \left(1 + \frac{1}{2}\right)\cos(2x) + \left(\frac{1}{2} + \frac{1}{4}\right)\cos(3x).$$

Problem 11: Calculate the following improper integrals. If they diverge say so.

(1)
$$\int_{-2}^{2} \frac{3x^2 + 4x - 3}{\sqrt{|x(x-1)(x+3)|}}\,dx$$

Answer: Notice that the integral is improper at $x = 0$ and at $x = 1$. Break it into three integrals:

$$\int_{-2}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx$$

The middle integral is improper for two different reasons. Exam the first integral first. Notice that for $x \in [-2,0]$, $x(x-1)(x+3) \geq 0$. Hence:

$$\int_{-2}^{0} f(x)\,dx = \int_{-2}^{0} \frac{3x^2+4x-3}{\sqrt{x^3+2x^2-3x}}\,dx$$

A simple substitution shows that this is equal to $\int_6^0 \frac{du}{\sqrt{u}} = -2\sqrt{6}$.
Rewrite the middle integral as

$$\int_{0}^{1/2} f(x)\,dx + \int_{1/2}^{1} f(x)\,dx$$

In this range $x(x-1)(x+3) \leq 0$. Thus to get rid of the absolute value introduce a - sign under the square root.

$$\int_{0}^{1/2} \frac{3x^2+4x-3}{\sqrt{-x(x-1)(x+3)}}\,dx + \int_{1/2}^{1} \frac{3x^2+4x-3}{\sqrt{-x(x-1)(x+3)}}\,dx$$

These can both be solved by substitution giving:

$$\int_{0}^{7/8} \frac{-du}{\sqrt{u}}\,du + \int_{7/8}^{0} \frac{-du}{\sqrt{u}}\,du$$

which equals 0.
The third integral can be rewritten as:

$$\int_{1}^{2} \frac{3x^2+4x-3}{\sqrt{x(x-1)(x+3)}}\,dx$$

Solving this by substitution yields:

$$2\sqrt{u}\Big|_{0}^{10} = 2\sqrt{10}$$

Thus the answer to the integral is $2\sqrt{10} - 2\sqrt{6}$.

(2)

$$\int_{-\infty}^{0} \frac{1}{2x-5}\,dx$$

Answer: This integral diverges to $-\infty$.

(3)

$$\int_{-1}^{1} \frac{e^x}{e^x - 1}\, dx$$

Answer: This integral diverges.

(4)

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$$

Answer: π

Problem 12: Determine whether the following integrals converge or diverge. Be sure to give a complete explanation.

(1)

$$\int_{0}^{\pi} \sin\left(\frac{1}{x}\right) dx$$

Answer: Converges

(2)

$$\int_{0}^{1} \frac{\sin(x)}{x}\, dx$$

Answer: Converges. The function $y = \frac{\sin(x)}{x}$ is bounded on $[0, 1]$.

(3)

$$\int_{0}^{\infty} \frac{\cos^2(x)}{1+x^2}\, dx$$

Answer: The integrand is always positive and less than or equal to $\frac{1}{1+x^2}$. We know that $\int_0^\infty \frac{1}{1+x^2}\, dx$ converges by Problem 11.4.

(4)

$$\int_3^\infty \frac{\ln(x^2+2)}{x}\,dx$$

Answer:

$$\begin{aligned}
\int_3^\infty \frac{\ln(x^2+2)}{x}\,dx &\geq \int_3^\infty \frac{\ln(x^2)}{x}\,dx \quad \text{since } \ln \text{ is an increasing function} \\
&= \int_3^\infty \frac{2\ln(x)}{x}\,dx \\
&\geq \int_3^\infty \frac{2}{x}\,dx \qquad \text{since } x \geq e, \ln x \geq 1
\end{aligned}$$

This last integral diverges to infinity as we saw in class. Since all the functions involved are non-negative, the comparison theorem shows us that our integral also diverges to infinity.

Problem 13: Prove that if $f(x)$ is a continuous function such that $\int_0^\infty |f(x)|\,dx$ converges then $\int_0^\infty f(x)\,dx$ also converges.

Answer: Define the following functions:

$$f_+(x) = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$f_-(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Notice that $f = f_+ - f_-$ and $|f| = f_+ + f_-$. Both f_+ and f_- are positive functions with $f_+(x) \leq |f(x)|$ and $f_-(x) \leq |f(x)|$ for all values of x. We can, therefore apply the comparison theorem to conclude:

$$\begin{aligned}
\int_0^\infty f_+(x)\,dx &\leq \int_0^\infty |f(x)| < \infty \\
\int_0^\infty f_-(x)\,dx &\leq \int_0^\infty |f(x)| < \infty
\end{aligned}$$

That is, both integrals converge. Since the difference of two convergent integrals converges we have:

$$\int_0^\infty f(x)\,dx = \int_0^\infty f_+(x) - f_-(x)\,dx = \int_0^\infty f_+(x)\,dx - \int_0^\infty f_-(x)\,dx < \infty.$$

Problem 14: Consider the graph of $f(x) = x^{-3/2}$ for $x \in [1, \infty)$.

(1) Determine whether or not the solid obtained by rotating the region between the graph and the x axis around the x axis has finite volume.

Answer: Using the disc method, the area of the solid is $\int_1^\infty \pi(x^{-3/2})^2\,dx$. This is equal to $\int_1^\infty x^{-3}\,dx$. Rewrite as:

$$\begin{aligned} \lim_{s\to\infty} \int_1^s x^{-3}\,dx &= \lim_{s\to\infty} -\frac{1}{2}x^{-2}\Big|_1^s \\ &= \lim_{s\to\infty} -\frac{1}{2}\left(\frac{1}{s^2} - 1\right) \\ &= \frac{1}{2} \end{aligned}$$

So the volume is finite.

(2) Determine whether or not the solid obtained by rotating the region between the graph and the x axis around the y axis has finite volume.

Use the shell method to find the volume is $\int_1^\infty 2\pi x(x^{-3/2})\,dx$. Rewrite this as:

$$\begin{aligned} 2\pi \lim_{s\to\infty} \int_1^s x^{-1}2\,dx &= 2\pi \lim_{s\to\infty} 2\sqrt{x}\Big|_1^s \\ &= 2\pi \lim_{s\to\infty}(2\sqrt{s} - 2) \\ &= \infty \end{aligned}$$

And so the volume is infinite.

Problem 15: The normal curve is the function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for $x \in (-\infty, \infty)$. Show that the area between the graph of $f(x)$ and the x-axis is finite.

Answer: Write the area times $\sqrt{2\pi}$ as:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx \quad = \quad \int_{-\infty}^{-1} e^{-x^2/2}\,dx + \int_{-1}^{1} e^{-x^2/2}\,dx + \int_{1}^{\infty} e^{-x^2/2}\,dx$$

Consider these three integrals individually. The middle one is just a definite integral of a continuous function and so is a finite number. To show the total area is finite we need to show that the first and third integrals converge.

In fact, since $y = e^{-x^2/2}$ is an even function, the first and third integrals are equal. Therefore, it suffices to consider the third integral:

$$\begin{aligned}
\int_{1}^{\infty} e^{-x^2/2}\,dx &\leq \int_{1}^{\infty} e^{-x/2}\,dx \\
&= \lim_{s\to\infty}(-2)(e^{-x/2})\Big|_{1}^{s} \\
&= -2(0 - e^{-1/2}) \\
&= 2e^{-1/2}
\end{aligned}$$

Thus, the comparison theorem shows that our integral converges. We do, however, need to explain why the first inequality holds. To see this notice that for $x \geq 1$: $-x^2 \leq -x$. This means that $-x^2/2 \leq -x/2$. Since the exponential function is an increasing function, $e^{-x^2/2} \leq e^{-x/2}$. The comparison theorem does the rest.