Math 10: Some Sequences Converging to the Golden Ratio

Example 1: Ratios of Fibonacci numbers Let $\{f_k\}$ be the Fibonacci Sequence. Recall that $f_{k+1} = f_k + f_{k-1}$. Define:

$$a_k = \frac{f_{k+1}}{f_k}$$

Question: Assuming that $L = \lim_{k\to\infty} a_k$ exists, figure out what it must be.

Notice that:

$$a_k = \frac{f_{k+1}}{f_k} = 1 + \frac{f_{k-1}}{f_k}$$

So that

$$L = 1 + \lim_{k\to\infty} \frac{f_{k-1}}{f_k}$$

Notice that $\frac{1}{a_{k-1}} = \frac{f_{k-1}}{f_k}$. This means that

$$L = 1 + \frac{1}{L}$$

Solving for L gives $L = \frac{1\pm\sqrt{5}}{2}$. Since every a_k is positive, we can discard the negative version, obtaining:

$$L = \frac{1+\sqrt{5}}{2}$$

Example 2: An infinite square root

Define $a_1 = 1$ and $a_k = \sqrt{1 + a_{k-1}}$. We show that $\lim_{k\to\infty} a_k$ exists and equals the golden ratio.

Part 1: $1 \leq a_k \leq 2$

Notice that each $a_k \geq 1$ since a_k^2 is bigger than 1.

Notice that $a_1 < 2$. Suppose that $a_{k-1} < 2$. Then

$$1 + a_{k-1} < 3$$

Which means

$$\sqrt{1+a_{k-1}} < \sqrt{3} < 2.$$

Thus, if $a_{k-1} < 2$, so is a_k. Since $a_1 < 2$, so is a_2. Since $a_2 < 2$ so is a_3. Continuing we see that every $a_k < 2$. Thus the sequence is bounded.

Part 2: $\{a_k\}$ is increasing.

Notice first of all that $a_2 > a_1$. Now suppose that $a_k > a_{k-1}$. This means that:

$$\frac{1+a_k}{a_{k-1}} > 1 + \frac{1}{a_{k-1}}$$

That is,

$$1 + a_k > a_{k-1} + 1$$

Which means:

$$\sqrt{1+a_k} > \sqrt{1+a_{k-1}}$$

That is:

$$a_{k+1} > a_k$$

Thus, if $a_k > a_{k-1}$ we also have $a_{k+1} > a_k$. Since $a_2 > a_1$ we then have $a_3 > a_2$ and then $a_4 > a_3$. Continuing on, we see that we have an increasing sequence.

Since the sequence $\{a_k\}$ is an increasing, bounded sequence it must converge to a limit L.

Part 3: Find L

Write $L = \sqrt{1+\sqrt{1+\sqrt{1+\dots}}}$ and notice that $L = \sqrt{1+L}$. Solving for L we discover:

$$L = \frac{1+\sqrt{5}}{2}$$

Example 3: A continued fraction Define $a_1 = 1$ and let $a_k = 1 + \frac{1}{a_{k-1}}$. We will show that $L = \lim_{k\to\infty} a_k$ exists and equals the golden ratio.

Part 1: The sequence is bounded above

Notice that each $a_{k-1} \geq 1$ since it is 1 plus a positive number. Hence, $\frac{1}{a_{k-1}} \leq 1$ and so $1 + \frac{1}{a_{k-1}} \leq 2$. The left hand side is the definition of a_k and so $a_k \leq 2$.

Part 2: Is the sequence monotonic?

Unfortunately, no. You can show that if $a_k \leq a_{k-1}$ then $a_{k+1} \geq a_k$.

We cannot apply the theorem to this sequence. However you might try:

Extra-Credit: Show that $\lim_{k\to\infty} a_k$ exists.

Hint: Show that the sequences $\{a_{2k}\}$ and $\{a_{2k+1}\}$ each converge using the theorem (one is decreasing, the other increasing). Then use the ϵ-definition of convergence and that result to show that $\{a_k\}$ converges. You will need to use our work from the next part to show that $\{a_{2k}\}$ and $\{a_{2k+1}\}$ each converge to the same number, which is the golden ratio. See me for help if you wish.

Part 3: Supposing that $\lim_{k\to\infty} a_k$ exists, find it. Write:

$$L = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}}$$

Notice that $L = 1 + \frac{1}{L}$. Solve for L to find that:

$$L = \frac{1+\sqrt{5}}{2}.$$