

Math 19: Practice Final

Name:

You are allowed an $8\frac{1}{2}'' \times 11''$ sheet of paper (both sides) for the exam. You may use a calculator, but not its graphing or calculus features. This practice exam is longer than the actual exam, though the problems are of similar difficulty.

- (1) Find the parametric equation of the line which is orthogonal to the plane $3x - 2y + z = 6$ and passes through the point $(-1, -1, -1)$.

- (2) Let \mathbf{u} and \mathbf{v} be vectors in \mathbb{R}^2 . State and prove the formula for the vector projection of \mathbf{u} onto \mathbf{v} .

- (3) Let $\mathbf{r}(t)$ be the space curve $(t \sin t, \sin t, t)$. Find the unit tangent, unit normal, unit binormal vectors to \mathbf{r} at the point when $t = 2\pi$. Also find the curvature of $\mathbf{r}(t)$ when $t = 0$.

- (4) Prove that if $\|\mathbf{r}(t)\| = 1$ for all t then $\mathbf{r}'(t)$ and $\mathbf{r}(t)$ are orthogonal for all t .

- (5) Compute the length of the curve $\mathbf{r}(t) = \begin{pmatrix} \frac{1}{2}t \\ \ln t \\ 2\sqrt{t} \end{pmatrix}$ for $1 \leq t \leq e$.
- (6) Find all critical points $0 \leq x \leq \pi$, $0 \leq y$ of the function $f(x, y) = y \sin x - \frac{1}{2}y$. Determine whether each is a max, min, or saddle.
- (7) Let $f(x, y) = y \sin x - \frac{1}{2}y$. Let $\mathbf{u} = \frac{1}{\sqrt{2}}(1, -1)$. Find the directional derivative of $f(x, y)$ in the \mathbf{u} direction at the point $(0, \pi)$.
- (8) Prove that $\nabla f(p, q)$ is the direction from (p, q) in which $f(x, y)$ increases fastest.
- (9) Let $f(x)$ and $g(x)$ be continuous functions. Use a $\delta - \epsilon$ proof to prove that $f(x) + g(x)$ is a continuous function.
- (10) Prove that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 + y^4}{x^2 + y^2}$ equals 0. (This is another $\delta - \epsilon$ proof).

- (11) Prove that $\lim_{(x,y) \rightarrow (0,0)} \frac{x+y^2}{x+2y^2}$ does not exist.
- (12) Give an example of a function $f(x, y)$ which has partial derivatives with respect to both x and y but which is not differentiable at $(0, 0)$.
- (13) Prove that $f(x, y) = xy$ is differentiable at $(0, 0)$ using the limit definition of the derivative.

- (14) Let S be the surface given by the parameterization:

$$\mathbf{r}(u, v) = \begin{pmatrix} \sin u \\ \cos v \\ v \end{pmatrix} \text{ for } 0 \leq u \leq \pi/2 \text{ and } 0 \leq v \leq \pi/2$$

Consider the vector field:

$$F(x, y, z) = \begin{pmatrix} x \\ y^2 \\ \sin z \end{pmatrix}$$

Compute the surface integral $\iint_S F \, d\mathbf{S}$.

- (15) Calculate the area enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Do this by using the change of coordinates $x = 2u$ and $y = 3v$. Be sure you explain each step and use the change of coordinates formula for double integrals correctly.

(16) Let B be the portion of the ball of radius 4 between $z = 0$ and $z = 2$. Use a triple integral in spherical coordinates to calculate the volume of B . (Hint: Split the integral into two pieces. One where $\pi/3 \leq \phi \leq \pi/2$ and $0 \leq \rho \leq 4$ and the other where $0 \leq \phi \leq \pi/3$ and $0 \leq \rho \leq 2 \sec \phi$.)

(17) Let C be the line segment in the plane which joins $(0,1)$ to $(3,2)$. Let $f(x,y) = xe^y$. Compute $\int_C f(x,y) \, d\mathbf{r}$.

(18) Prove that if $F(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$ is a conservative vector field then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

(19) Let $F(x,y) = \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \end{pmatrix}$. Show that every two positively oriented curves containing the origin produce the same line integral of F .

(20) Let S be the portion of the paraboloid $z = x^2 + y^2 - 4$ which is between $z = 0$ and $z = 12$. Use Stoke's Theorem to compute $\int_C F \, d\mathbf{r}$ where C is the positively oriented boundary of S and

$$F(x,y,z) = \begin{pmatrix} y \\ -z \\ x \end{pmatrix}.$$

Hint: S can be parameterized as $\mathbf{r}(u, v) = \begin{pmatrix} u \\ v \\ u^2 + v^2 \end{pmatrix}$ for $-4 \leq u \leq 4$ and $-\sqrt{16 - u^2} \leq v \leq \sqrt{16 - u^2}$.

(21) Is the vector field $F(x, y, z) = \begin{pmatrix} xe^y \\ ye^z \\ ze^x \end{pmatrix}$ a conservative vector field? Give a good reason for your answer.

(22) Consider the path C parameterized by $\mathbf{r}(t) = \begin{pmatrix} \cos t \\ \tan t \\ \sin t \end{pmatrix}$ for $0 \leq t \leq \pi/4$. Let

$$F(x, y, z) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$$

be a force field. The field F is conservative. Compute the amount of work necessary to move a 1 kg object along the path C .

(23) This problem is related to the proof of Green's Theorem; You are not allowed to use Green's theorem in its solution. Let D be the square in the xy plane with vertices $(0, 0)$, $(4, 0)$, $(4, 3)$, $(0, 3)$ and let C be its positively oriented boundary. Let

$F(x, y) = \begin{pmatrix} P \\ Q \end{pmatrix}$ be a C^1 vector field. Show that:

$$\int_C P \, dx = - \iint_D \frac{\partial P}{\partial y} \, dA$$

- (24) Recall that if $\int_C F \, d\mathbf{r} = 0$ for every simple closed curve C then the vector field $F(x, y, z)$ is conservative. Use this fact and Stoke's theorem to prove that if $\text{curl } F = \mathbf{0}$ then F is conservative.

- (25) Let S be the cylinder with boundary given by the equations $x^2 + y^2 = 4$, $z = -4$ and $z = +4$. Use the divergence theorem to compute $\iint_S F \, d\mathbf{S}$ where F is the vector field

$$F(x, y, z) = \begin{pmatrix} x^2 \\ y^4 \\ xyz \end{pmatrix}.$$

(Hint: You may want to compute the triple integral using cylindrical coordinates.)