The Big Theorems of Vector Calculus

1. Definitions

Scalar Field: A scalar field is simply a function $f : \mathbb{R}^2 \to \mathbb{R}$ or $f : \mathbb{R}^3 \to \mathbb{R}$. An example is a function which assigns a temperature to each point of space.

Vector Field: A vector field is simply a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ or $F : \mathbb{R}^3 \to \mathbb{R}^3$. An example is the velocity field of a moving fluid. A **gradient field** can be obtained by letting $F = \nabla f$ for any scalar field f.

Conservative Vector Field: A vector field F is conservative if there is a scalar field f so that $F = \nabla f$.

Parameterized Curve: A parameterized curve is a (differentiable) function $r : \mathbb{R} \to \mathbb{R}^2$ or $r : \mathbb{R} \to \mathbb{R}^3$.

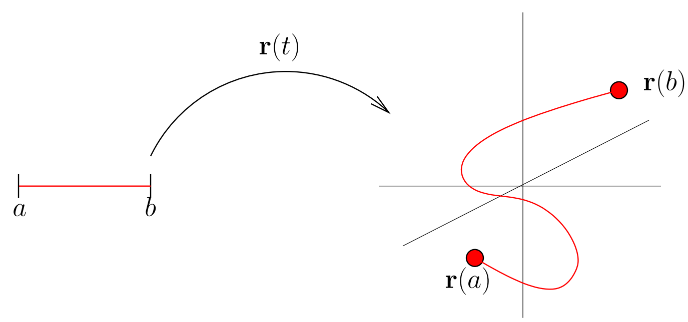

Figure 1. A parameterized curve

Parameterized Surface: A parameterized surface is a (differentiable) function $r : \mathbb{R}^2 \to \mathbb{R}^3$.

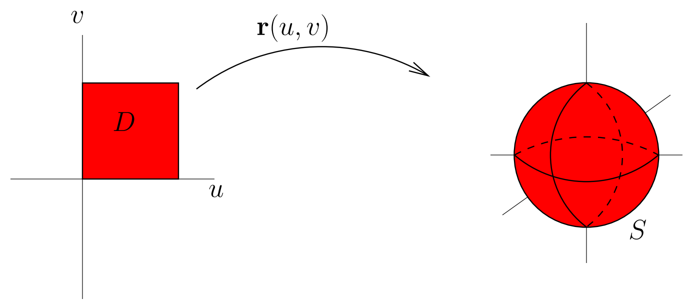

FIGURE 2. A parameterized surface

Line Integral of a Scalar Field:

$$\int_C f \ d\mathbf{s} = \int_a^b f(\mathbf{r}(t))||\mathbf{r}'(t)|| \ dt$$

Line Integral of a Vector Field:

$$\int_C F \ d\mathbf{r} = \int_a^b F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \ dt$$

Surface Integral of a Scalar Field:

$$\iint_S f \ dS = \iint_D f(\mathbf{r}(u,v)||\mathbf{r}_u \times \mathbf{r}_v|| \ dudv$$

Surface Integral of a Vector Field:

$$\iint_S f \ d\mathbf{S} = \iint_D F(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \ dudv$$

2. Important Theorems

Fundamental Theorem of Calculus
If $F = \nabla f$ is a conservative vector field then for any curve C that starts at A and ends at B we have $\int_C F\, d\mathbf{r} = f(B) - f(A)$.

Green's Theorem

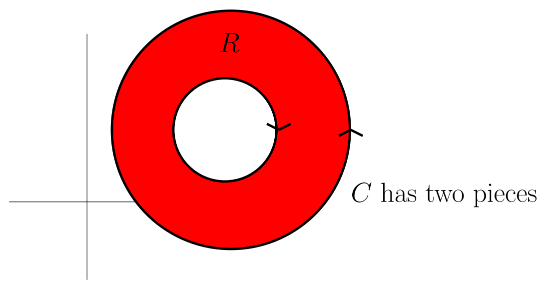

Figure 3. The region R

If
$$F(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$$
is a C^1 vector field and if C is the **positively oriented** boundary of a region R then:

Version 1:
$$\int_C F\, d\mathbf{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

Version 2:
$$\int_C F\, d\mathbf{r} = \iint_R (\operatorname{curl} F) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\, dA$$

Version 3:
$$\int_C F \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} F\, dA$$
where $\mathbf{n}$ is the unit normal to C.

Stoke's Theorem

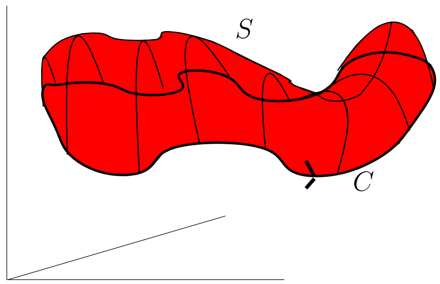

FIGURE 4. The surface S

If $F(x, y, z)$ is a C^1 vector field and if C is the positively oriented boundary of an orientable surface S in $\mathbb{R}^3$ then:

$$\int_C F \, d\mathbf{r} = \iint_S \operatorname{curl} F \, d\mathbf{S}$$

Divergence Theorem

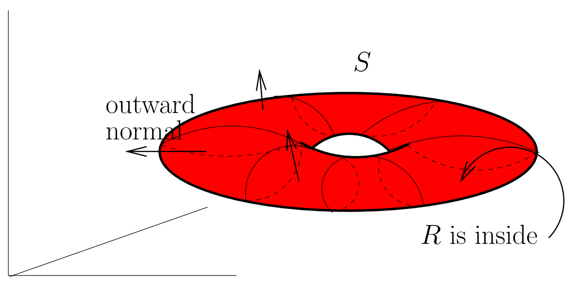

FIGURE 5. The surface S and region R

If $F(x, y, z)$ is a C^1 vector field and if S is a surface which is the boundary of a region R and if S has outward pointing normal vectors then:

$$\iint_S F \, d\mathbf{S} = \iiint_E \operatorname{div} F \, dxdydz$$