Some Answers to Exam 2

Name:

(4) Find all critical points in $[0, \pi) \times [0, \pi)$ of the function:

$$f(x, y) = \frac{1}{4}\sin(2x)\sin(2y)$$

and determine whether each is a maximum, minimum, or saddle.

Solution: Critical points occur when $\nabla f(x, y) = 0$. We have:

$$\begin{aligned} f_x(x, y) &= \frac{1}{2}\cos(2x)\sin(2y) \\ f_y(x, y) &= \frac{1}{2}\sin(2x)\cos(2y) \end{aligned}$$

The first equation tells that $2x = \frac{\pi}{2}, \frac{3\pi}{2}$ or that $2y = 0, \pi$. That is either we have $x = \frac{\pi}{4}, \frac{3\pi}{4}$ or $y = 0, \frac{\pi}{2}$. Notice that all these points are in the domain of x and y.

The second equation tells us that $2y = \frac{\pi}{2}, \frac{3\pi}{2}$ or that $2x = 0, \pi$. That is either we have $y = \frac{\pi}{4}, \frac{3\pi}{4}$ or $x = 0, \frac{\pi}{2}$. Notice that all these points are in the domain of x and y.

To find critical points we get to pick one of x or y from solutions to the first equation and the other one from solutions to the second equation. Therefore we get the following critical points (x, y):

A. $(0, 0)$
B. $(0, \pi/2)$
C. $(\pi/2, 0)$
D. $(\pi/2, \pi/2)$
E. $(\pi/4, \pi/4)$
F. $(\pi/4, 3\pi/4)$
G. $(3\pi/4, \pi/4)$
H. $(3\pi/4, 3\pi/4)$

To determine if these are maximima, minima, or saddles we need to look at the Hessian, the matrix of 2nd partials/

$$\begin{aligned}
f_{xx}(x,y) &= -\sin(2x)\sin(2y)\\
f_{xy}(x,y) &= \cos(2x)\cos(2y)\\
f_{yx}(x,y) &= \cos(2x)\cos(2y)\\
f_{yy}(x,y) &= -\sin(2x)\sin(2y)
\end{aligned}$$

So at each critical point we have the following Hessians:

A.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

B.

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

C.

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

D.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

E.

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

F.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

G.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

H.

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Finally, calculate the determinate and trace of each Hessian. If the determinate is negative we have a saddle. If it is positive, the trace being positive tells us we have a minimum and the trace being negative tells us we have a maximum.

Crit Pt	Det	Trace	Type
A	-1	0	Saddle
B	-1	0	Saddle
C	-1	0	Saddle
D	-1	0	Saddle
E	1	-2	Max
F	1	2	Min
G	1	2	Min
H	1	-2	Max

(9) Let R be the object defined by the following bounds:

$$\begin{array}{ccccc} 0 & \leq & y & \leq & 1 \\ y & \leq & x & \leq & \sqrt{1+y^2} \\ \ln(y^2+y) \leq z \leq \ln(y^2+y)+3y & & & & \end{array}$$

Use a triple integral to comput the volume of R. Hint: Consider the change of variables:

$$\begin{array}{rcl} x & = & \sqrt{u^2+v^2} \\ y & = & v \\ z & = & \ln(v^2+v)+vw \end{array}$$

Solution: We will use the change of variables formula. To do that we need to compute the bounds of the corresponding region in terms of u, v, w and we need to compute the determinate of the jacobian of the transformation. The change of coordinates formula is:

$$\int\int\int_R f(x,y,z)dxdydz = \int\int\int_{R^*} f(u,v,w)|detDT|dudvdw$$

First compute the new region R^*:

$$\begin{array}{ccccc} 0 & \leq & v & \leq & 1 \\ v & \leq & \sqrt{u^2+v^2} & \leq & \sqrt{1+v^2} \\ \ln(v^2+v) & \leq & \ln(v^2+v)+vw & \leq & \ln(v^2+v)+3v \end{array}$$

The idea is now to do some simple algebra to rewrite the bounds so that u, v, w are in the middle of their respective inequalities. **There is one issue here that needs to be addressed (though it didn't affect the grading). We'll discuss it at the end of the solution.

After some algebra we get:

$$\begin{array}{ccccc} 0 & \leq & v & \leq & 1 \\ 0 & \leq & u & \leq & 1 \\ 0 & \leq & w & \leq & 3 \end{array}$$

Now we can find $|detDT|$. To do that write down the matrix of partial derivatives from the equations for the transformation:

$$DT = \begin{pmatrix} u(u^2+v^2)^{-1/2} & v(u^2+v^2)^{-1/2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2v+1}{v^2+v}+w & v \end{pmatrix}$$

Thus $|detDT| = \frac{uv}{\sqrt{u^2+v^2}}$. Now we can write down the integral:

$$\int_0^3 \int_0^1 \int_0^1 \frac{uv}{\sqrt{u^2+v^2}} dudvdw.$$

The inside integral can solved by substitution $r = u^2$. We'll use this substitution again later:

$$\begin{array}{cc} \int_0^3 \int_0^1 \int_0^1 \frac{uv}{\sqrt{u^2+v^2}} dudvdw & = \\ \int_0^3 \int_0^1 \frac{v}{2} \int_0^1 (r+v^2)^{-1/2} drdvdw & = \\ \int_0^3 \int_0^1 v\sqrt{1+v^2} - v^2 dvdw & = \\ \int_0^3 \int_0^1 \frac{1}{2}\sqrt{1+s}ds - \frac{1}{3}v^3|_0^1 dw & = \\ \int_0^3 \frac{1}{2}\frac{2}{3}(1+s)^{3/2}|_0^1 - \frac{1}{3}dw & = \\ \int_0^3 \frac{1}{3}(2)^{3/2} - \frac{1}{3}dw - 1 & = \\ \sqrt{8} - 2 & \end{array}$$

**Now for the tricky bit. Notice that in the original bounds for R, y was allowed to equal 0. In the bounds for z, however, we have $\ln(y^2+y)$ which is undefined when $y = 0$. Here's a way around that:

Let R_c for $1 > c > 0$ be the object with bounds:

$$\begin{array}{ccccc} c & \leq & y & \leq & 1 \\ y & \leq & x & \leq & \sqrt{1+y^2} \\ \ln(y^2+y) & \leq & z & \leq & \ln(y^2+y)+3y \end{array}$$

The volume of R is then equal to $\lim_{c\to 0^+} \int\int\int_{R_c} dxdydz$. Notice that this solves the division by 0 problem in calculating the limits for u. Now go through the above work. Your final answer will have c in it. Now take the limit as $c \to 0^+$. You'll get the answer above.