Math 19: Practice Exam 2

Name:

What follows are solutions, partial solutions, or hints for the problems on the Practice Exam. In particular, on the actual exam you will need to show and explain all of your work.

(1) Show (using the limit definition of the derivative) that the function $f(x,y) = x(y-1)^2$ is differentiable at the point $(0,0)$.

Answer: The linear approximating function is $L(x,y) = x$. We will use the squeeze theorem. Other approaches may also work.

$$0 \leq \lim_{(x,y)\to(0,0)} \frac{|f(x,y)-L(x,y)|}{||(x,y)-(0,)||} =$$

$$\lim_{(x,y)\to(0,0)} \frac{|x(y-1)^2-x|}{\sqrt{x^2+y^2}} =$$

$$\lim_{(x,y)\to(0,0)} \frac{|x||(y-1)^2-1|}{\sqrt{x^2+y^2}} \leq$$

$$\lim_{(x,y)\to(0,0)} \frac{|x||y||y-2|}{\sqrt{x^2}} =$$

$$\lim_{(x,y)\to(0,0)} \frac{|x||y||y-2|}{|x|} =$$

$$\lim_{(x,y)\to(0,0)} |y||y-2| = 0$$

(2) How quickly does the function $f(x,y) = \ln(x^2 + y^2)$ increase or decrease at (1,1) in the direction $\mathbf{u} = \frac{1}{\sqrt{20}}(-4, 2)$.

Answer: We have $\nabla f(1,1) = (1,1)$. The formula for the directional derivative in the direction of $\mathbf{u}$ is:

$$f_{\mathbf{u}}(1,1) = \nabla f(1,1) \cdot \mathbf{u} = \frac{-2}{\sqrt{20}}.$$

(3) Find all critical points in $[0, 2\pi] \times [-4, 4]$ of the function:

$$f(x, y) = \sin(x) + \frac{1}{3}y^3 - 4y + \cos(x)$$

and determine whether each is a maximum, minimum, or saddle.

Answer: The two critical points are $(\pi/4, 2)$ and $(\pi/4, -2)$. The first is a saddle and the second a maximum.

(4) Let $f(x, y) = x^3 y$. Approximate the volume contained between the graph of $f(x, y)$ and the rectangle $R = [0, 3] \times [0, 3]$. Use $n = 3$ subintervals of each interval. Use the upper left hand corner of each subrectangle as the x_{ij}^*.

Answer: $V \approx 54$.

(5) Let $f(x,y) = \frac{xy}{x^2+y^2}$. Let R be the region in the xy plane between the curves $y = x$, $y = 3x$, $x = 0.5$, and $x = 3$. Compute $\int\int_R f(x,y)dA$.

You may wish to use:

$$\int \ln u du = u \ln u - u + C$$

Answer:
Fubini's theorem tells us the given integral is:

$$\int_{0.5}^{3} \int_{x}^{3x} \frac{xy}{x^2+y^2} dydx$$

Perform a substitution on the inner integral to get:

$$\int_{0.5}^{3} \frac{x}{2} \int_{x^2}^{9x^2} \frac{1}{x^2+u} dudx$$

This equals:

$$\int_{0.5}^{3} \frac{1}{2} x(\ln(10x^2) - \ln(2x^2)dx$$

Perform another substitution to get:

$$\frac{1}{4}\int_{0.25}^{9} \ln(10u)du - \frac{1}{4}\int_{0.25}^{9} \ln(2u)du$$

Performing two last substitutions get:

$$\frac{1}{40}\int_{2.5}^{90} \ln(v)dw - \frac{1}{8}\int_{.5}^{18} \ln(w)dw$$

Do the integrations:

$$\frac{1}{40}[90(\ln(90)-1)-2.5(\ln(2.5)-1)]+\frac{1}{8}[18(\ln(18)-1)-0.5(\ln(0.5)-1)]$$

(6) Use polar coordinates to compute:

$$\int\int_D \sin\left(\frac{x^2+y^2}{2}\right) \; dxdy$$

D is the region pictured below:

Answer:

The integral is

$$\int_{\pi/4}^{3\pi/4} \int_0^4 \sin\left(\frac{r^2}{2}\right) r dr d\theta$$

(7) Let E be the tetrahedron bounded by the planes $z = 0$, $x = 0$, $y = 0$, $x+2y+z = 1$. Suppose that its density can be described by the function $\rho(x, y, z) = e^{x-y+z}$. Compute its mass.

Answer: The integral is:

$$\int_0^1 \int_0^{\frac{1}{2}-\frac{1}{2}x} \int_0^{1-x-2y} e^{x-y+z} dz dy dx$$

(8) Let R be the object defined by the following bounds:

$$\begin{array}{ccccc} 1 & \leq & x & \leq & 2 \\ \ln(x) & \leq & y & \leq & e^x \\ \sqrt{x+y} & \leq & z & \leq & \sqrt{x+y+2} \end{array}$$

Use a triple integral with the change of variables:

$$\begin{array}{ccc} x & = & u \\ y & = & \ln v \\ z & = & \sqrt{w+\ln v+u} \end{array}$$

to write the volume of R as a single definite integral of a function of u.

Answer: The answer is:

$$\int_1^2 \frac{2}{3}(2+e^u+u)^{3/2} - \frac{2}{3}(e^u+u)^{3/2} - \frac{2}{3}(2+2u)^{3/2} + (2u)^{3/2} du$$

(9) Use cylindrical coordinates to compute the volume of the region defined by:

$$\begin{array}{rcccl} \arcsin(y/\sqrt{x^2+y^2}) & \leq & z & \leq & \arccos(x/\sqrt{x^2+y^2}) + x^2 + y^2 \\ -x & \leq & y & \leq & x \\ -1 & \leq & x & \leq & 1 \end{array}$$

Answer: The bounds become:

$$\begin{array}{rcccl} \theta & \leq & z & \leq & \theta + r^2 \\ -1 & \leq & \tan\theta & \leq & 1 \\ -1 & \leq & r\cos\theta & \leq & 1 \end{array}$$

The second inequality is equivalent to $-\pi/4 \leq \theta \leq \pi/4$. The third inequality is then equivalent to $-\sec\theta \leq r \leq \sec\theta$. The determinate of the Jacobian is $r dr d\theta$. Thus our integral is:

$$\int_{-\pi/4}^{\pi/4} \int_{-\sec\theta}^{\sec\theta} \int_{\theta}^{\theta+r^2} r dz dr d\theta$$

Finally, compute the integral:

$$\int_{-\pi/4}^{\pi/4} \int_{-\sec\theta}^{\sec\theta} r^3 dr d\theta = 0$$

(10) You will also be asked one of the following questions:

Answer: You'll need to look these up in the book or in your notes.

(a) Prove that for a unit vector $\mathbf{u}$ and a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f_{\mathbf{u}}(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

(b) Prove that ∇f is the direction in which f increases most quickly.

(c) Suppose that $g : \mathbb{R} \to \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ are differentiable functions. Explain why the chain rule $(f(g(x)))' = f'(g(x)) \cdot g'(x)$ works.

(d) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ which has partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial g}{\partial y}$ at $(0, 0)$ but which is not differentiable at $(0, 0)$.