The Algebra of Transformations

These exercises explore some elementary algebraic operations that correspond to the transformations we have studied in class. **Use graph paper for all graphs.** You may either paste a section of graph paper into your main paper or you may do the graphs on a separate sheet.

Underlying this work is the idea of the Cartesian Plane. In order to keep things simple, we mostly will stick with integers. These initial problems should be review. If you cannot remember how to do them, consult pages 41-55 in Krause.

1. Plot the following points on a single graph labeled **Graph I.** Be sure to label each point with its letter \((a, b, c, d)\).
   a. \((0, 0)\)  b. \((-1, 2)\)  c. \((2, -4)\)  d. \((-2, 0)\)

   **Vector addition:** We can view these points as vectors and add them using the definition \((a, b) + (c, d) = (a + c, b + d)\). For example, \((1, 3) + (-2, 3) = (1 + -2, 3 + 3) = (-1, 6)\) and \((2, -1) + (0, -1) = (2 + 0, -1 + (-1)) = (2, -2)\).

2. Perform the following computations on your main homework sheet.
   (a) \((1, 3) + (2, -1) = \)
   (b) \((-2, 4) + (0, -1) = \)
   (c) \((1, 3) + (0, 0) = \)
   (d) \((1, 3) + (-1, -3) = \)
   (e) \((1, 3) + \underline{\phantom{0}} = (0, -1)\)
   (f) \((1, 3) + \underline{\phantom{0}} = (-2, 4)\)
   (g) \((-2, 4) + \underline{\phantom{0}} = (0, -1)\)

   (Extra Credit: comment on any interesting algebraic ideas related to what you just did.)

3. Do the following for each of the points below:
   a. \((0, 0)\)  b. \((-1, 2)\)  c. \((2, -4)\)  d. \((-2, 0)\)

   (a) Add \((1, 2)\) to the point.
   (b) Plot the resulting point on **Graph I** and label with \(a', b', c', d'\).
(c) Draw arrows from each original point to its corresponding prime.

Comment on what you observe. What transformation has been performed?

4. Describe in geometric terms (translation, rotation, reflection, etc.) what would happen if you added \((2, 4)\) to each of the points \(a, b, c, d\) in 3.

5. Give an algebraic description (addition, subtraction, etc.) of the transformation that will translate the plane so that the point \((-4, 2)\) is moved to the point \((-1, 1)\). In other words, what algebra should be done to an arbitrary point \((x, y)\) in the plane?

**WARNING!** In the following section we are using a nonstandard definition of matrix multiplication. Usually the vector would be oriented as \(
\begin{bmatrix} a \\ b \end{bmatrix}
\) rather than as \((a, b)\). Since we will be using only a very restricted portion of matrix multiplication, no harm will come from using this nonstandard method for this brief exercise.

Definition: We define the product of a matrix \(
\begin{bmatrix} a & b \\ c & d \end{bmatrix}
\) and a vector \((x, y)\) to be the vector \((ax+by, cx+dy)\). For example, \(
\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}
\) \((2, 3) = (1 \cdot 2 + 0 \cdot 4, 4 \cdot 2 + 5 \cdot 3) = (2, 23)\) and \(
\begin{bmatrix} 0 & -2 \\ 2 & 1 \end{bmatrix}
\) \((-1, 3) = (0 \cdot -1 + -2 \cdot 3, 2 \cdot -1 + 1 \cdot 3) = (-6, 1)\).

You can often simplify your work by noticing the structure of the particular matrix multiplication. For example, multiplication by the first matrix above will always keep the same \(x\) value as the original. The second matrix multiplies the \(y\) coordinate by -2 and moves it to the \(x\) position.

6. For each of the four matrices below do the following:

\[
\begin{align*}
(1) & \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & (2) & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & (3) & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & (4) & \begin{bmatrix} .7071 & -.7071 \\ .7071 & .7071 \end{bmatrix}
\end{align*}
\]

Make a graph for each of the four matrices. Label the graphs \textbf{Graph II(1)} through \textbf{Graph II(4)}. On each graph:

(a) Multiply the four points \(a, b, c, d\) from problem 3 by the given matrix to create points \(a', b', c', d'\). For all but (4) the work can be drastically reduced by checking what happens to a generic point \((x, y)\).
(b) Plot and label the original four points $a, b, c, d$ from problem 3 and connect the three points $b, c, d$ into a figure using straight lines.

(c) Plot and label the four computed $a', b', c', d'$ and connect the three points $b', c', d'$ into a figure using straight lines.

(d) Identify the transformation. (e.g. “reflection across the $x$-axis,” “clockwise rotation by $30^\circ$ around the origin.”)

**Isometries based at points other than (0,0):** You should have noticed that in all of these transformations, the point $(0, 0)$ was fixed. In particular, all the rotations were rotations about the point $(0, 0)$ and that all the reflections were reflections through lines that passed through $(0,0)$. For example, matrix (2) produces a $90^\circ$ clock-wise rotation about the origin $(0,0)$.

If we want to rotate around a point $(\alpha, \beta)$ other than $(0,0)$, we can translate that point to the origin, rotate, and then translate $(0,0)$ back to $(\alpha, \beta)$.

**Example 1** To rotate the plane $90^\circ$ clock-wise about the point $(2, 3)$ we would

(a) Translate so that $(2, 3)$ goes to the origin.

(b) Rotate the plane $90^\circ$ clock-wise about the origin.

(c) Translate so that $(0, 0)$ goes to $(2, 3)$.

(To understand how this works, try drawing a figure and the point $(2, 3)$ on a plain sheet of paper and move the paper according the to directions.)

7. For the following two isometries, follow the form in **Example 1** and write out the three geometric steps that would

(a) rotate the plane by $90^\circ$ clock-wise about the point $(-1, 2)$,

(b) flip the plane over the line $y = x + 2$. (You can move any point on the line to the origin.)

**Turning the individual steps into algebra:** The individual geometric steps can each be turned into algebraic operations.

**Example 2** To rotate the plane about the point $(2, 3)$ we would

(a) Subtract $(2, 3)$ from each vector.
(b) Multiply each vector by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(c) Add $(2, 3)$ to each vector.

8. For the two isometries in 7, identify the algebraic operations corresponding to each of the individual geometric steps you identified in 7. Follow the form of Example 2.

**Combining and simplifying the algebraic operations:** A sequence of operations like those in exercise 8 can be collapsed into a single algebraic formula by composing the operations.

**Example 3** Suppose that we want to construct a single algebraic formula that will tell us where to send an arbitrary point $(x, y)$ when the plane is rotated $90^\circ$ clock-wise around the point $(2, 3)$.

(i) First translate so $(2, 3)$ goes to $(0, 0)$:

$$(x, y) \mapsto (x, y) - (2, 3) = (x - 2, y - 3)$$

(ii) Rotate/reflect by the appropriate matrix:

$$(x - 2, y - 3) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (x - 2, y - 3) = (y - 3, -x + 2)$$

(iii) Translate so $(0, 0)$ goes to $(2, 3)$:

$$(y - 3, -x + 2) \mapsto (y - 3, -x + 2) + (2, 3) = (y - 1, -x + 5)$$

So, rotation by $90^\circ$ clock-wise around the point $(2, 3)$ is given by the single algebraic operation of

$$(x, y) \mapsto (y - 1, -x + 5)$$

9. For each of the two isometries from 7, combine the individual algebraic operations that you identified in 8 to produce a single algebraic operation. Follow the form of Example 3.

10. Apply your two algebraic expressions in 9 to the four points $a, b, c, d$ of 3. Illustrate the isometry by plotting both the original points and their images $a', b', c', d'$ on two graphs labeled **Graph IIIa** and **Graph IIIb**

**Some additional background information for your general edification and education:** Although we will not do so, it is possible to rewrite the result algebraic result of this three-step process as a matrix multiplication followed by a translation. In fact, any isometry can be written using a single matrix multiplication and a single translation. This is an algebraic parallel to
the result derived in class that any isometry can be expressed as the product of three or fewer reflections.

We can write out an algebraic expression for any of the three types of basic isometries (reflection, translation and rotation) that we have studied. You already know how to express translations (which have the simplest algebraic form). Rotation by \( \theta \) degrees around the origin is accomplished by the matrix

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

Reflections, the simplest isometries in “Mira Land,” have the most complicated expressions in “Algebra Land.” Reflection across the line through the origin oriented at \( \theta \) degrees to the x-axis is given by the matrix

\[
\begin{bmatrix}
1 - 2\sin^2 \theta & 2 \cos \theta \sin \theta \\
2 \cos \theta \sin \theta & 1 + 2\sin^2 \theta
\end{bmatrix}
\]

These origin-centered transformations can be applied at other points using the three step techniques developed earlier. All of these algebraic representations of transformations extend quite naturally to higher dimensions and are used in computer graphics applications such as drawing programs and CAD/CAM programs.